The Gauss Circle Problem

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Outline

Introduction

- The Gauss circle problem is the problem of determining the number of integer lattice points inside the circle of radius r centered at the origin. Let $Q(r)$ be the number of lattice points inside a circle in plane of radius r, i.e.

$$N(r) = \#\{(m,n) \in \mathbb{Z}^2 \mid m^2 + n^2 \leq r^2\} \ .$$

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- Hence the real problem is to accurately bound $E(r)$. The goal is to find a bound of the form

$$|E(r)| = O\left(r^{\theta}\right)$$

for θ as small as possible.

Historical bounds

θ	approx.	Name	Year
1	1	Gauss	1834
2/3	0.66667	Voronoi Sierpinski van der Corput	1903 1906 1923
37/56	0.66071	Littlewood and Walfisz	1925
27/41	0.65854	van der Corput	1928
35/54	0.64813	Kolesnik	1982
34/53	0.64151	Vinogradov	1935
7/11	0.63636	Iwaniec and Mozzochi	1988
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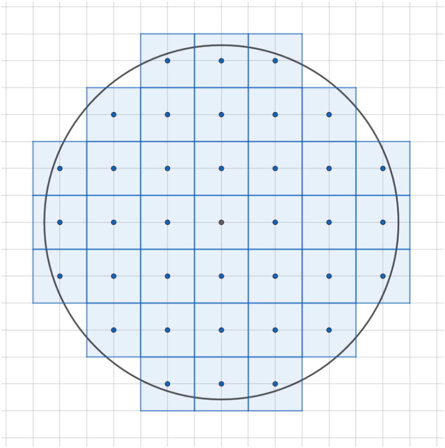
Regarding lower bounds, we know that $\limsup_{x\to\infty} \frac{|E(x)|}{x^{1/2}(\log x)^{1/4}} > 0$ (Hardy, 1915). It is conjectured that $E(r) = O\left(r^{1/2+\epsilon}\right)$, for all $\epsilon > 0$.

Proof by picture



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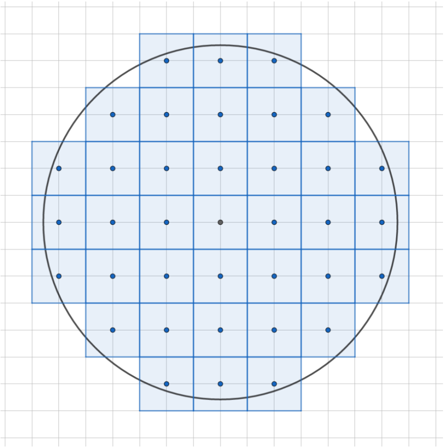

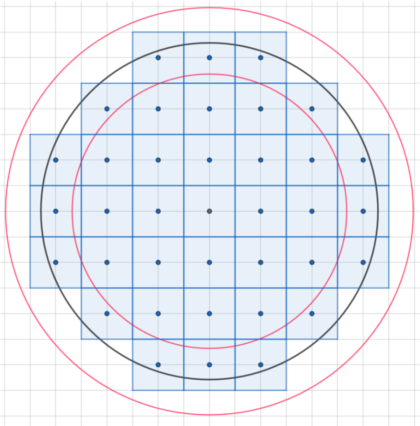

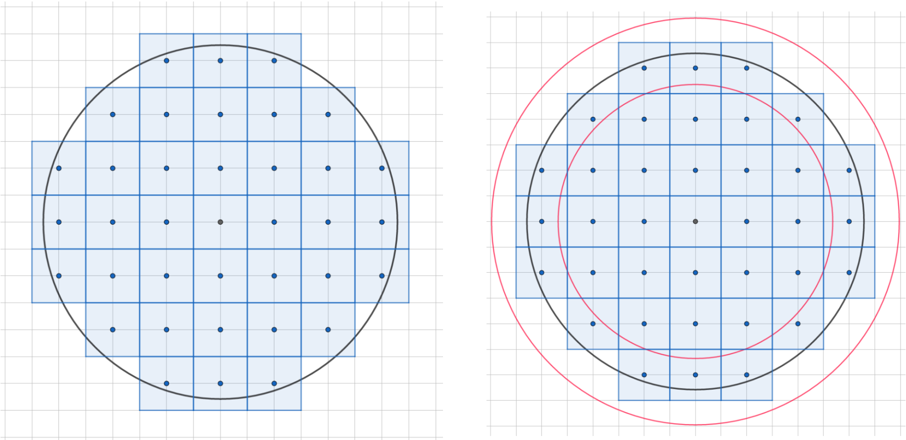

We have that $\pi(r-\sqrt{2}/2)^2 \leq N(r) \leq \pi(r+\sqrt{2}/2)^2$, so $|E(r)| \leq \sqrt{2}\pi r + \pi/2$ (hence $E(r) = O(r)$).

Classical exponent - Introduction

Let $\mathbf{1}(r)$ be the characteristic function of the unit disc in plane, i.e.

$$\mathbf{1}(\mathbf{x}) = \begin{cases} 1 & \text{if } |\mathbf{x}| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

and let $\mathbf{1}_r(\mathbf{x}) = \mathbf{1}(\mathbf{x}/r)$ the characteristic function of the disc of radius r. Then

$$N(r) = \sum_{\mathbf{x} \in \mathbb{Z}^2} \mathbf{1}_r(\mathbf{x})$$

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Now let ρ be a positive smooth function on $\mathbb{R}^2$ with compact support inside the unit ball and integral one. Also, define

$$\rho_\epsilon(\mathbf{x}) = \frac{1}{\epsilon^2} \rho\left(\frac{\mathbf{x}}{\epsilon}\right)$$

so that ρ_ϵ is supported inside the ball of radius ϵ and has integral still equal to 1. Next define

$$N'_\epsilon(r) = \sum_{\mathbf{x} \in \mathbb{Z}^2} (\mathbf{1}_r * \rho_\epsilon)(\mathbf{x})$$

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$$\begin{aligned} (\mathbf{1}_r * \rho_\epsilon)(\mathbf{x}) &= 1 \quad \text{if } |\mathbf{x}| \leq r - \epsilon \\ (\mathbf{1}_r * \rho_\epsilon)(\mathbf{x}) &= 0 \quad \text{if } |\mathbf{x}| > r + \epsilon \\ 0 \leq (\mathbf{1}_r * \rho_\epsilon)(\mathbf{x}) &\leq 1 \quad \text{if } r - \epsilon \leq |\mathbf{x}| \leq r + \epsilon. \end{aligned}$$

Therefore $N'_\epsilon(r - \epsilon) \leq N(r) \leq N'_\epsilon(r + \epsilon)$.

Classical exponent - Poisson summation formula

Theorem (Poisson summation formula)

If $f \in S(\mathbb{R}^n)$, *then* $\sum_{\mathbf{x}\in\mathbb{Z}^n} f(\mathbf{x}) = \sum_{\mathbf{x}\in\mathbb{Z}^n} \widehat{f}(\mathbf{x})$.

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- Next we notice that

$$\widehat{\mathbf{1}}_r(\xi) = \int_{\mathbb{R}^2} \mathbf{1}_r(\mathbf{x})e(-\mathbf{x}.\xi)\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^2} \mathbf{1}(\mathbf{y})e(-r\mathbf{y}.\xi)r^2\,\mathrm{d}\mathbf{y} = r^2\,\widehat{\mathbf{1}}(r\xi)$$

$$\widehat{\rho_\epsilon}(\xi) = \int \rho_\epsilon(\mathbf{x})e(-\mathbf{x}.\xi) = \int \rho(\mathbf{y})e(-\epsilon\mathbf{y}.\xi) = \hat{\rho}(\epsilon\xi)$$

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- Since $\widehat{\mathbf{1}}(0) = \pi$ and $\widehat{\rho}(0) = 1$, we have that

$$N'_\epsilon(r) = \pi r^2 + r^2 \sum_{\mathbf{x}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \widehat{\mathbf{1}}(r\mathbf{x})\widehat{\rho}(\epsilon\mathbf{x})$$

Estimates for $\widehat{1}$ and $\widehat{\rho}$

- We have that $\left|\widehat{\mathbf{1}}(\mathbf{x})\right| \ll |\mathbf{x}|^{-3/2}$, for $|\mathbf{x}| \geq 1$. Indeed,

$$\widehat{\mathbf{1}}(\mathbf{x}) = \int_{|\mathbf{y}|\leq 1} e^{-2\pi i(\mathbf{x}.\mathbf{y})}\, \mathrm{d}\mathbf{y} = \frac{J_1(2\pi|\mathbf{x}|)}{|\mathbf{x}|}$$

where J_1 is the Bessel function of the first kind

$$J_1(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(y-\sin y)}\, \mathrm{d}y$$

The estimate follows from the asymptotic formula

$$J_1(|\mathbf{x}|) = \sqrt{\frac{2}{\pi|\mathbf{x}|}}\left(\cos(|\mathbf{x}| - \frac{3\pi}{4}) + O(|\mathbf{x}|^{-1})\right)$$

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- Now, since ρ is smooth with compact support, then $\rho \in S(\mathbb{R}^2)$. This implies that also $\widehat{\rho} \in S(\mathbb{R}^2)$. In particular, $|\widehat{\rho}(\mathbf{x})| \ll_N (1 + |\mathbf{x}|^2)^{-N}$, for all positive integers N.

Finish proof of the classical exponent

- Now we approximate the error term:

$$r^2 \sum_{\mathbf{x}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \widehat{\mathbf{1}}(r\mathbf{x})\widehat{\rho}(\epsilon\mathbf{x}) \ll r^{1/2} \sum_{\mathbf{x}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} |\mathbf{x}|^{-3/2}(1+\epsilon^2|\mathbf{x}|^2)^{-2}$$

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- Next we take $\epsilon = r^{-1/3}$, so we have that $N'_\epsilon(r) = \pi r^2 + O(r^{2/3})$. Hence

$$N'_\epsilon(r+\epsilon) = \pi(r+r^{-1/3})^2 + O((r+r^{-1/3})^{2/3}) = \pi r^2 + O(r^{2/3})$$

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- We obtain a similar estimate for $N'_\epsilon(r-\epsilon)$. Hence

$$N(r) = \pi r^2 + O(r^{2/3}) \; .$$

The method of exponent pairs

- We want to to find upper bounds for

$$S = \sum_{n=A}^{B} e(f(n))$$

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- We would like to find an upper bound of the form

$$S \ll L^k N^l$$

  for $0 \leq k \leq 1/2 \leq l \leq 1$. If this bound holds for all "nice" functions f, we say (k, l) is an *exponent pair*.

A and B processes

- A process: If (k, l) is an exponent pair, then so is

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  (easy to see that $B^2(k,l) = (k,l)$).
- The method consists of deriving new exponent pairs of the form $A^{q_1}BA^{q_2}B\ldots A^{q_k}B(0,1)$ or $BA^{q_1}BA^{q_2}B\ldots A^{q_k}B(0,1)$ and use them to bound exponential sums.

Definitions

Definition

Let N, P, y, s, ϵ be positive numbers with $\epsilon < 1/2$. We define $\mathbf{F}(N, P, s, y, \epsilon)$ to be the set of functions f with P continuous derivatives on I such that for all $0 \leq p \leq P-1$ and $A \leq x \leq B$

$$\left| f^{(p+1)}(x) - (-1)^p (s)_p y x^{-s-p} \right| \leq \epsilon (s)_p y x^{-s-p} \tag{1}$$

where $(s)_0 = 1$ and $(s)_p = s(s+1)\dots(s+p-1)$ for $p \geq 1$.

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Let

$$F(x) = \begin{cases} \frac{yx^{1-s}}{1-s} & \text{if } s \neq 1 \\ y \log x & \text{if } s = 1 \end{cases} \tag{2}$$

Then (1) can be rewritten as

$$\left| f^{(p+1)}(x) - F^{(p+1)}(x) \right| \leq \epsilon \left| F^{(p+1)}(x) \right| \tag{3}$$

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$$|S| \ll_{k,l,s} (yN^{-s})^k N^l + y^{-1} N^s \tag{4}$$

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It is not hard to show that if $L = yN^{-s} \leq 1$, then we obtain satisfactory estimates in elementary ways . Hence the real strength of the method is in the case $L \geq 1$, where the main term is indeed $L^k N^l$.

Application to the Gauss Circle problem -preliminary lemmas

Lemma

$$N(r) = \pi x^2 + 4 \sum_{d \le x/4} \left(\psi\left(\frac{x^2}{4d+3}\right) - \psi\left(\frac{x^2}{4d+1}\right) + \psi\left(\frac{x^2}{4d} - \frac{3}{4}\right) - \psi\left(\frac{x^2}{4d} - \frac{1}{4}\right) \right) + O(1)$$

where $\psi(x) = \{x\} - 1/2$.

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Lemma

Say (k, l) *is an exponent pair and let* P *and* ϵ *be the corresponding parameters given by the definition of exponent pairs. If* $f \in \mathbf{F}(N, P, s, y, \epsilon)$, *then*

$$\left| \sum_{n \in I} \psi(f(n)) \right| \ll y^{\frac{k}{k+1}} N^{\frac{(1-s)k+l}{k+1}} + y^{-1} N^s$$

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- After completing the calculations, we obtain that if (k, l) is an exponent pair,

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- Now take

$$(k, l) = BA^3B(0, 1) = \left(\frac{11}{30}, \frac{26}{30}\right) \quad .$$

Then we have that $E(x) \ll x^{27/41}$, where $27/41 = 0.6585\ldots$

References

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