NON-VANISHING OF GEODESIC PERIODS OF AUTOMORPHIC FORMS

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Abstract. We prove that one hundred percent of closed geodesic periods of a Maaß form for the modular group are non-vanishing when ordered by length. We present applications to the non-vanishing of central values of Rankin–Selberg L-functions. Similar results for holomorphic forms for general Fuchsian groups of the first kind with a cusp are also obtained, as well as results towards normal distribution.

1. Introduction

The study of closed geodesics on the modular surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ is a rich and important subject, at the confluence of arithmetic, geometry, and dynamics. In particular, the closed geodesics encode deep arithmetic information via Waldspurger's formula. More precisely, let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant $D > 0$. For each element $A \in \mathrm{Cl}_K^+$ in the (narrow) class group, we can associate a closed geodesic $\mathscr{C}_A \subset \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ of length $2\log\epsilon_D$ (see e.g [32]). We know from the class number formula and a result of Siegel that $|\mathrm{Cl}_K^+|\log\epsilon_D = D^{1/2+o(1)}$. Let f be a Hecke–Maaß eigenform of weight 0 and level 1. Then we have by Waldspurger's formula due in its explicit form to Popa [29]:

$$L(f\otimes\theta_\chi, 1/2) = \frac{c_f^+}{D^{1/2}}\left|\sum_{A\in\mathrm{Cl}_K^+}\chi(A)\int_{\mathscr{C}_A} f(z)\frac{|dz|}{y}\right|^2 \tag{1.1}$$

where $\chi \in \widehat{\mathrm{Cl}_K^+}$ is a class group character, θ_χ is the associated theta series, $L(f\otimes\theta_\chi, s)$ is the Rankin–Selberg L-function of f and θ_χ, and $c_f^+ > 0$ is some constant depending only on f.

The present paper is concerned with the study of the arithmetic statistics of the geodesic periods $\int_{\mathscr{C}_A} f(z)\frac{|dz|}{y}$ and their generalization to general Fuchsian groups and general automorphic forms. More precisely, our work was motivated by the following question posed by Michel [5, p. 1377]:

Question 1.1 (Michel). *Fix $\delta > 0$ and a Hecke–Maaß form f for $\mathrm{PSL}_2(\mathbb{Z})$. Let K be a real quadratic field of discriminant D such that $h(D) \geq D^\delta$. For D large enough,*

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does there always exist $A \in \mathrm{Cl}_K^+$ such that $\int_{\mathscr{C}_A} f(z)\frac{|dz|}{y} \neq 0$? Equivalently, does there exist a (narrow) class group character $\chi : \mathrm{Cl}_K^+ \to \mathbb{C}^\times$ such that $L(f \otimes \theta_\chi, 1/2) \neq 0$?

Note that the equivalence between the two non-vanishing questions follows by Waldspurger's formula (1.1) and character orthogonality. The motivation for this question is the work of Michel–Venkatesh [21] in the imaginary quadratic analogue. In this case the geodesic periods corresponds to evaluating the Maaß form at Heegner points. Michel–Venkatesh combined subconvexity with equidistribtion of Heegner points (as proved by Duke [10]) to obtain non-vanishing result for the corresponding Rankin–Selberg L-functions. They also noticed that the corresponding argument using equidistribution of geodesics (also proved by Duke) falls short ultimately due to the existence of fundamental units for real quadratic fields. For more details see Section 2.

In this paper we introduce a new method which answers an average version of Michel's question in a strong sense. In particular, we show that when ordered by length, then 100% of the geodesic periods are non-vanishing. More precisely, let

$$C(X) := \{\mathscr{C} \subset \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H} : \ell(\mathscr{C}) \leq X\},$$

denote the set of primitive, closed geodesics on the modular curve with length $\ell(\mathscr{C}) := \int_{\mathscr{C}} 1\frac{|dz|}{y}$ bounded by X. From the prime geodesic theorem, with the best error term to date given by [35], we know

$$|C(X)| = \mathrm{Li}(e^X) + O\left(e^{X(25/36+\epsilon)}\right) = \frac{e^X}{X}(1 + o(1)), \quad \text{when } X \to \infty.$$

Our first main result is the following quantitative bound for the vanishing set.

Theorem 1.2. *Let f be a Maaß form for the modular group. Then*

$$\left|\left\{\mathscr{C} \in C(X) : \int_{\mathscr{C}} f(z)\frac{|dz|}{y} = 0\right\}\right| \ll \frac{e^X}{X^{5/4}}.$$

Remark 1.3. We obtain a similar result for holomorphic forms of weight $k \in 2\mathbb{Z}$, we refer to Theorem 6.1 for the more general statement. We also bound the size of geodesics with small or large geodesic period.

1.1. **Result for general Fuchsian groups.** More generally, given a Fuchsian group of the the first kind $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$, we have an associated finite volume hyperbolic 2-orbifold $X_\Gamma := \Gamma\backslash\mathbb{H}$. Let f be a either a Maaß form, holomorphic cusp form or a completed Eisenstein series $E^*_{\mathfrak{a},k}(z, \frac{1}{2} + it)$ for Γ and denote by $F : \Gamma\backslash\mathrm{PSL}_2(\mathbb{R}) \to \mathbb{C}$ the lift of f to the unit tangent bundle of X_Γ. The primitive oriented closed geodesics on X_Γ are in one-to-one correspondence with primitive hyperbolic conjugacy classes of Γ. Given an oriented closed geodesic $\mathscr{C}$ we can lift it canonically to the unit tangent bundle (which we will denote by the same symbol). We equip $\mathscr{C}$ with the unique A-invariant measure $\mu_{\mathscr{C}}$ which projects to the line element $\frac{|dz|}{y}$ on X_Γ (here

A denotes the diagonal subgroup of $\mathrm{PSL}_2(\mathbb{R})$). We refer to Section 3 for more details. A key quantity of study are the geodesic periods

$$\mathscr{P}_f(\mathscr{C}) := \int_{\mathscr{C}} F(g) d\mu_{\mathscr{C}}(g). \tag{1.2}$$

A natural question is to fix a closed geodesic and ask how the sizes of the geodesic periods behave as the Maaß form f is varying, see [34], [31], [4], [22] for results in this direction. In this paper we are interested in obtaining strong lower bound for the set of non-vanishing geodesic periods of a fixed automorphic form. To quantify this, put for $X \geq 1$:

$$C_\Gamma(X) := \{\mathscr{C} \subset \Gamma\backslash\mathbb{H} \text{ primitive closed geodesic} : \ell(\mathscr{C}) \leq X\}. \tag{1.3}$$

The first general result in this direction seems to be the work of Katok [18] who proved that for a holomorphic form f on Γ (including the co-compact case) there exists at least one non-vanishing geodesic period using Poincaré series techniques. Zelditch [41, Thm. 0.4] used a variant of the Selberg trace formula to estimate the first moment of the geodesic periods (averaged over $C_\Gamma(X)$) which yields non-vanishing when the Laplacian on X_Γ admits an eigenfunction with eigenvalue $0 < \lambda < \frac{19}{400}$. In the case of weight 2 holomorphic forms Petridis–Risager [27] used spectral methods to prove that the geodesic periods for co-compact Γ when ordered by geodesic length are normally distributed (and in particular 100% of them are non-vanishing).

Our second main result is the following non-vanishing theorem in the presence of a cusp.

Theorem 1.4. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind with a cusp. Let f be an automorphic form for Γ such that either*

- $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ *and f is a Maaß form, or*
- *f is a holomorphic cusp form of weight $k \in 2\mathbb{Z}_{>0}$.*

Then 100% of the geodesic periods of f are non-vanishing when ordered by length. More precisely, for any function $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $h(x) \to \infty$ as $x \to \infty$ it holds that

$$\frac{\left|\left\{\mathscr{C} \in C_\Gamma(X) : h(X)^{-1} \leq |\mathscr{P}_f(\mathscr{C})| / \sqrt{X} \leq h(X)\right\}\right|}{|C_\Gamma(X)|} \to 1, \tag{1.4}$$

as $X \to \infty$.

We have a similar result for Eisenstein series for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

Theorem 1.5. *Let $E_{k,t}(z) := E_k(z, 1/2 + it)$ be the Eisenstein series of weight k for $\mathrm{PSL}_2(\mathbb{Z})$. For any function $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $h(x) \to \infty$ as $x \to \infty$ it* *holds that*

$$\frac{\left|\left\{\mathscr{C} \in C_\Gamma(X) : h(X)^{-1} \leq \left|\mathscr{P}_{E_{k,t}}(\mathscr{C})\right| / (\sqrt{X}(\log X)^{3/2}) \leq h(X)\right\}\right|}{|C_\Gamma(X)|} \to 1, \tag{1.5}$$

as $X \to \infty$.

Restricting our Theorem 1.4 to the case of Hecke congruence groups and using a result of Raulf [30], we obtain the following non-vanishing theorem, which is an averaged version of the results of Michel–Venkatesh [21].

Corollary 1.6. *Let f be Hecke–Maaß form (holomorphic or non-holomorphic) of weight $k \in 2\mathbb{Z}$ and level $N \geq 1$ which is either holomorphic or of level 1. Then we have that as $X \to \infty$:*

$$\frac{|\{D \in \mathscr{D}^+_{\text{fund}} : \epsilon_D \leq X, \exists \chi \in \widehat{\mathrm{Cl}^+_D} \text{ s.t. } L(1/2, f \otimes \theta_\chi) \neq 0\}|}{|\{D \in \mathscr{D}^+_{\text{fund}} : \epsilon_D \leq X\}|} \geq c + o(1),$$

for some $c > 0$ (depending only on N). Here $\mathscr{D}^+_{\text{fund}}$ denotes the set of positive fundamental discriminants and $\epsilon_D > 0$ denotes the positive fundamental unit of discriminant D.

Remark 1.7. An interesting feature of our methods is that equidistribution does indeed play a key role exactly as in the arguments of Michel–Venkatesh: the proof of Theorem 1.4 relies crucially on equidistribution of sparse subcollections of closed geodesics, see Theorem 5.8.

Remark 1.8. We note that some "non-smallness" assumption is necessary in Question 1.1. When the narrow class number of $K = \mathbb{Q}(\sqrt{D})$ is one (which conjecturally should happen infinitely often) and the Hecke–Maaß form is odd then we have for the unique class group character χ_K of K that

$$L(f \otimes \theta_{\chi_K}, 1/2) = L(f, 1/2)L(f \otimes \chi_D, 1/2) = 0,$$

where χ_D denotes the quadratic Dirichlet character associated to K by class field theory.

1.2. **Towards normal distribution.** By Theorem 1.4 we see that the geodesic periods associated to $\mathscr{C}$ are usually of size $\ell(\mathscr{C})^{1/2}$. A related phenomena has previously been observed in the context of vertical periods (e.g. modular symbols) of automorphic forms [28] and the two phenomena are intimately related as we will see.

Let Γ be a Fuchsian group with a cusp at infinity of width one. Let f be a cusp form for Γ of weight $k \in 2\mathbb{Z}$. For $[\gamma]_\infty \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty$, we can associate the vertical line period integral:

$$[\gamma]_\infty \mapsto \int_{\gamma\infty}^{\infty} f(z)dz =: L_f(\gamma\infty).$$

If c_γ denotes the lower-left entry of the matrix γ, note that c_γ is invariant in $[\gamma]_\infty$ and put

$$T_\Gamma(N) := \{[\gamma]_\infty \in \Gamma_\infty\backslash\Gamma/\Gamma_\infty : 0 < |c_\gamma| \leq N\}$$

The distribution of $\{L_f(\gamma\infty) : [\gamma]_\infty \in T_\Gamma(N)\}$ has been extensively studied, it is known in many cases to obey asymptotically a normal distribution with variance $c_f^2 \log N$ as $N \to \infty$, see [2], [9], [28], [7], [23], [19].

In this paper we introduce a method to "lift" the distribution of the vertical periods to study the geodesic periods. In particular, this means that out methods relies crucially on the existence of a cusp for Γ. We do this by investigating the relation between the sets of double cosets $\Gamma_\infty\backslash\Gamma/\Gamma_\infty$ and hyperbolic conjugacy classes of Γ via the study of a particular graph. Recall that to each closed geodesic $\mathscr{C}$ on X_Γ corresponds a primitive hyperbolic conjugacy classes $\{\gamma\}$ of Γ such that the length $\ell(\mathscr{C})$ is up to a small error given by $\log|\operatorname{tr}(\gamma)|$. This means that

$$C_\Gamma(\log N) \approx \{\{\gamma\} \in \operatorname{Conj}(\Gamma) : 2 < |\operatorname{tr}(\gamma)| \leq N, \{\gamma\} \text{ primitive}\},$$

with $C_\Gamma(X)$ defined as in equation (1.3). We consider the bipartite graph G_N with the two vertex sets given by $T_\Gamma(N)$ and $C_\Gamma(\log N)$, and an edge between $[\gamma]_\infty \in T_\Gamma(N)$ and $\{\gamma\} \in C_\Gamma(\log N)$ if and only if $[\gamma]_\infty \cap \{\gamma\} \neq \emptyset$. There is a natural way to define a discrete probability measure μ_{G_N} on $C_\Gamma(\log N)$ induced via the graph G_N from the counting measure on $T_\Gamma(N)$ defined in equation (4.1), see Section 4 for more details. We show that the geodesic periods become normally distributed when counted with this measure:

Theorem 1.9. *Let f be a Maaß cusp form or holomorphic cusp form for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, or f a holomorphic cusp form for a general Fuchsian group Γ of the first kind with a cusp. Then there exists a constant $c_f > 0$ such that for any rectangle $\mathscr{R} \subset \mathbb{C}$, we have*

$$\lim_{N\to\infty} \mu_{G_N}\left(\left\{\{\gamma\} \in C_\Gamma(\log N) : \frac{\mathscr{P}_f(\{\gamma\})}{c_f\sqrt{\log N}} \in \mathscr{R}\right\}\right) = \frac{1}{2\pi}\int_{x+iy\in\mathscr{R}} e^{-\frac{x^2+y^2}{2}}\,dxdy,$$

where $\mathscr{P}_f(\{\gamma\}) = \mathscr{P}_f(\mathscr{C})$ denotes the geodesic periods associated to closed geodesic $\mathscr{C}$ corresponding to the conjugacy class $\{\gamma\}$.

In other words, geodesic periods are normally distributed if we count them with weights inferred by the graph G_N. We expect that the normal distribution should hold without these additional weights, i.e. for the counting measure. Note that Theorem 1.4 essentially states that most geodesic periods are distributed around the mean, as one would expect from normal distribution.

Conjecture 1.10. *Let f be a weight k Maaß form for a general Fuchsian group Γ. Then for any rectangle $\mathscr{R} \subset \mathbb{C}$,*

$$\frac{1}{|C_\Gamma(\log N)|}\left|\left\{\{\gamma\} \in C_\Gamma(\log N) : \frac{\mathscr{P}_f(\{\gamma\})}{c_f\sqrt{\log N}} \in \mathscr{R}\right\}\right|$$
$$= \frac{1}{2\pi}\int_{x+iy\in\mathscr{R}} e^{-\frac{x^2+y^2}{2}} dxdy + O\left(\frac{1}{\sqrt{\log N}}\right).$$

Remark 1.11. As alluded to above, in the case of weight 2 holomorphic forms for a cocompact Γ a weaker version of this conjecture (without the explicit rate of convergence) was proved by Petridis–Risager [26] by studying perturbations of the Selberg trace formula. In forthcoming work, using a different method as in [7], the authors are proving the full conjecture for weight 2 holomorphic forms for general Γ.

Remark 1.12. Note that the rate of convergence in this conjecture implies the size of the vanishing set is $\ll \frac{N}{(\log N)^{3/2}}$.

1.3. **Idea of proof.** As mentioned above, our main idea is to transfer the distribution (in particular, the non-vanishing) of vertical periods to geodesic periods. To illustrate this it is useful to consider the case where f is a weight 2 holomorphic cusp form. Then $f(z)dz$ is a Γ-invariant 1-form. Hence the following period, known as a modular symbols,

$$\langle \gamma, f\rangle := \int_z^{\gamma z} f(z)dz$$

does not depend on the base point z. Moreover, the map $\langle \cdot, f\rangle : \Gamma \to \mathbb{C}$ is additive. This implies that $\langle \gamma, f\rangle$ is invariant on both $[\gamma]_\infty$ and $\{\gamma\}$ and so $\langle \gamma, f\rangle$ is constant on connected components of the graphs G_N defined above. So in this case there is a straightforward connection between vertical line periods and geodesic periods.

For automorphic forms of general weight this is no longer true. We use instead a key ingredient from [24], which allows us to find a relation between the vertical line period and the geodesic period corresponding to an edge in the bipartite graph G_N defined above. In particular, we obtain that for 100% of edges vanishing of the geodesic period implies that the vertical period is very small. In the cases where the normal distribtution of vertical periods is known, we know that basically 100% of vertical periods are large (of size $\sqrt{\log N}$). This yields a contradiction if we can bound from above the degrees in G_N of vertices in $T_\Gamma(N)$ and bound from below the degrees of vertices in $C_\Gamma(\log N)$.

One can easily obtain good estimates for degrees in $T_\Gamma(N)$ from the basic properties of the graph (by counting matrices with fixed lower left entry and bounded trace). Using a geometric argument, we show that the degree of a vertex in $C_\Gamma(\log N)$ is lower bounded by the length of the corresponding geodes restricted to a subdomain of the fundamental domain for $\Gamma\backslash\mathbb{H}$. We show there exists a region $\mathscr{B} \subset \Gamma\backslash\mathbb{H}$ such that

$$\deg(\{\gamma\}) \gg l(\mathscr{C} \cap \mathscr{B}). \tag{1.6}$$

Following an approach of Aka–Einsiedler [1] we combine effective mixing [40] with an equidistribution theorem of Zelditch [41] to obtain an equidistribution theorem for sparse subcollections of closed geodesics, see Theorem 5.8, thus obtaining lower bounds for the right-hand side of (1.6) on average. This implies that the degrees of vertices of $C_\Gamma(\log N)$ on average are of the expected size which yields the wanted.

Remark 1.13. We obtain stronger non-vanishing results for the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, and the reason is twofold. Firstly, in this case, from [2] and [9], we know precise rate of convergence towards the normal distribution of the set $\{L_f(\gamma_\infty) : [\gamma]_\infty \in T_\Gamma(N)\}$, and hence we can deduce better upper bounds for the subset of cosets with small vertical line period. Secondly, in the arithmetic setting, it is possible to obtain equidistribution for a sparser subcollection, as in [1].

1.4. **Structure of paper.** In Section 2 we briefly discuss the work of Michel-Venkatesh for the imaginary quadratic case and why their methods fail in the real quadratic case.
In Section 3 we introduce the background material, including the connection between vertical line integral and geodesic periods from [24].
In Section 4 we develop the required graph theory, including measures on graphs and their lifts.
In Section 5 we look in more detail at the properties of the graph G_N defined above. We also prove here the sparse equidistribution results for geodesics, required to lower bound degrees on this graph.
In Section 6 we complete the proofs of our main Theorems 1.2 and 1.4.
In Section 7 we obtain the application towards non-vanishing of central values of Rankin-Selberg L-functions.

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2. Equidistirbution of Heegner points and non-vanishing of Rankin–Selberg L-functions, following Michel–Venkatesh

In an elegant paper [21] Michel–Venkatesh introduced a new method for evaluating the first moment of certain Rankin–Selberg L-functions associated to theta series of class group characters of imaginary quadratic fields. They approach used equidistribution of Heegner points combined with Waldspurger's formula and the Plancherel formula. This was combined with subconvexity to obtain quantitative non-vanishing results. The methods has subsequently been extended to calculating more general "wide moments" of L-functions by the second-named author [25], see also [6]. During the problem session at the Oberwolfach workshop [5, p. 1377] Michel asked whether one could extend this to real quadratic field under some "non-smallness" assumption on the class number (see Question 1.1 above). The difficulty of this problems stems from the infinite unit group which makes the original approach in [21] fall short as we will now explain.

Let f be a weight 0 Hecke–Maaß form of level 1. For an imaginary quadratic field $K/\mathbb{Q}$ of discriminant $D<0$ we denote by Cl_K the class group and $h(D)=\#\mathrm{Cl}_K$ the class number. Given $A\in\mathrm{Cl}_K$ we denote by $z_A\in X_0(1):=\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ the associated Heegner point (for a defintion see). Let $\chi\in\widehat{\mathrm{Cl}_K}$ be a class group character of K and denote by θ_χ the associated theta-series via automorphic induction. Waldspurger's formula due to Zhang [42], [43] in its explicit form is:

$$L(f\otimes\theta_\chi,1/2)=\frac{c_f}{D^{1/2}}\left|\sum_{A\in\mathrm{Cl}_K}\chi(A)f(z_A)\right|^2.$$

By Plancherel (i.e. character orthogonality) and Duke's equidistribution theorem for Heegner points [10] it follows that as $D\to-\infty$

$$\frac{1}{h(D)}\sum_{\chi\in\widehat{\mathrm{Cl}_K}}L(f\otimes\theta_\chi,1/2)=\frac{c_f}{D^{1/2}}\sum_{A\in\mathrm{Cl}_K}|f(z_A)|^2=\frac{c_fh(D)}{D^{1/2}}(\|f\|_{L^2}+o(1)).\tag{2.1}$$

In particular, the left-hand side is non-vanishing for $|D|$ sufficiently large. Furthermore, by applying the subconvexity bound for the Rankin–Selberg L-functions due to Harcos–Michel [13] one obtains a non-vanishing proportion of $\gg D^\delta$ for some small $\delta>0$.

Now let K be a real quadratic field of discriminant $D>0$ with narrow class group Cl_K^+. In this setting we can associate to an ideal class $A\in\mathrm{Cl}_K^+$ a primitive oriented closed geodesics $\mathscr{C}_A$ on $X_0(1)$, which are analogues to the association of Heegner points above. Let f be a weight zero Hecke–Maaß form of level 1. Similarly to the imaginary case, by employing the formula (1.1) due to Popa [29, Thm. 1] and orthogonality of characters we obtain:

$$\frac{1}{h(D)}\sum_{\chi\in\widehat{\mathrm{Cl}_K}}L(f\otimes\theta_\chi,1/2)=\frac{c_f^+}{D^{1/2}}\sum_{A\in\mathrm{Cl}_K}\left|\int_{\mathscr{C}_A}f(z)\tfrac{|dz|}{\mathrm{Im}\,z}\right|^2.\tag{2.2}$$

We see however that equidistribution of the geodesics does *not* imply non-vanishing of the right-hand side due to the fact that the square is on the "outside" of the integral (over the closed geodesics). To fix this, one has to allow for non-trivial infinity type and consider the Arakelov class group of K:

$$\mathrm{Cl}_K^{\mathrm{Ara}}:=K^\times\mathbb{A}_\mathbb{Q}^\times\backslash\mathbb{A}_K^\times/\widehat{\mathscr{O}}_K^\times\cong\mathrm{Cl}_K^+\times\mathbb{R}_{>0}/\epsilon_K^\mathbb{Z},$$

where $\epsilon_K>1$ is the positive fundamental unit of K. Given a character $\chi\in\widehat{\mathrm{Cl}_K^{\mathrm{Ara}}}$, the infinity type is parameterized by a number $\lambda_\chi\in\frac{1}{\log\epsilon_K}\mathbb{Z}$. As explained in [3, Sec.

4.2] using equation (4.7) in *loc. cit.* one has:

$$\frac{1}{h(D)} \sum_{\chi \in \widehat{\mathrm{Cl}_K^{\mathrm{Ara}}}} L(f \otimes \theta_\chi, 1/2)\psi_f(\lambda_\chi) = \frac{c_f^+}{D^{1/2}} \sum_{A \in \mathrm{Cl}_K^+} \int_{\mathscr{C}_A} |f(z)|^2 \tfrac{|dz|}{\mathrm{Im}\, z},$$

where $\psi_f(\lambda_\chi)$ denotes some weight function satisfying, by [3, (4.8)], the following bound:

$$\psi_f(\lambda_\chi) \ll e^{-c_0|\lambda_\chi|/|\lambda_f|},$$

for some absolute constant $c_0 \geq 0$. Notice that the equidistribution theorem of Duke yields that indeed the left-hand side is non-zero for D large enough. Thus we obtain non-vanishing Rankin-Selberg L-functions with χ an Arakelov class group character and by the decay property of $\psi_f(\lambda_\chi)$ we can ensure that $|\lambda_\chi| \ll (\log D)^2$, say. Notice that this is a family of characters of size $D^{1/2+o(1)}$ exactly as in the imaginary quadratic case. In other words, the question of Michel amounts to whether one can ensure that the infinity type is trivial.

Remark 2.1. In [21], similar results are obtained for holomorphic forms of weight 2 using the Jacquet–Langlands correspondence and Waldspurger's formula for definite quaternion algebras due to Gross in this case.

Remark 2.2. Templier extended the approach of Michel–Venkatesh to derivatives [37] using the Gross–Zagier formula. Later Templier [36] and Templier–Tsimerman [38] evaluated the left-hand side of (2.1) using tools from analytic number theory (approximate functional equation and (non-split) shifted convolution sums). See also [16]. It would be interesting to see whether a variation of these analytic methods can be made to work in the real quadratic case.

3. Background

3.1. **Fuchsian groups.** Let $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind and consider the associated quotient surface

$$X_\Gamma := \Gamma\backslash\mathbb{H}.$$

It is a general fact that there is a ono-to-one correspondence between primitive closed geodesics on X_Γ and primitive conjugacy classes of hyperbolic elements. We denote by $\mathscr{C}_\gamma$ the geodesic corresponding to the conjugacy class $\{\gamma\}$.

Each hyperbolic $\gamma \in \Gamma$ is conjugate in $\mathrm{PSL}_2(\mathbb{R})$ with some $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ with $t > 1$. The norm of γ is $N(\gamma) = t^2$, the trace is $\mathrm{tr}(\gamma) = t + t^{-1}$ and the length of corresponding geodesic is

$$\ell(\mathscr{C}_\gamma) := \int_{\mathscr{C}_\gamma} 1 \frac{|dz|}{y} = \log N(\gamma) = 2\log t.$$

The unit tangent bundle of X_Γ is naturally described as a symmetric space:

$$\mathbf{T}^1(X_\Gamma) \simeq \Gamma\backslash\mathrm{PSL}_2(\mathbb{R}),$$

recalling that $\mathbb{H} \simeq \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}_2$. The unit tangent bundle $\mathbf{T}^1(X_\Gamma)$ admits a right action of $\mathrm{PSL}_2(\mathbb{R})$ given by $\Gamma x.g = \Gamma xg$. The action of

$$a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad t \in \mathbb{R}$$

corresponds to the geodesic flow on $\mathbf{T}^1(X_\Gamma)$.

Consider the diagonal subgroup

$$A := \{a_t : t \in \mathbb{R}\} \leq \mathrm{PSL}_2(\mathbb{R}).$$

Let $\mathscr{C} \subset X_\Gamma$ be an oriented closed geodesic and consider its lift to the unit tangent bundle. By a slight abuse of notation, we denote the lift by the same symbol $\mathscr{C} \subset \mathbf{T}^1(X_\Gamma)$. This yields a one-to-one correspondence between oriented closed geodesic on X_Γ and closed and compact A-orbits in $\mathbf{T}^1(X_\Gamma)$.

We denote by $\mu_{\mathscr{C}}$ the unique A-invariant measure on $\mathscr{C}$ which descends to the line element $ds = \frac{|dz|}{\operatorname{Im} z}$ when $\mathscr{C}$ is projected to X_Γ. Let $F : \Gamma\backslash\mathrm{PSL}_2(\mathbb{R}) \to \mathbb{C}$ integrable with respect to $\mu_{\mathscr{C}}$. Then we have explicitly

$$\int_{\mathscr{C}} F(g) d\mu_{\mathscr{C}}(g) = \int_0^{\ell(\mathscr{C})} F(xa_t) dt, \quad \text{for some } x \in \mathscr{C}.$$

We refer to [12, Chapter 9] for more details about the geodesic flow on quotient surfaces.

3.1.1. *Spectral theory of automorphic forms.* We will now recall some standard facts about the spectral theory of automorphic forms, we refer to [11, Sec. 4] for further background. Let $f : \mathbb{H} \to \mathbb{C}$ be an *automorphic form of weight k for* Γ, that is a smooth function such that

$$f(\gamma z) = j_\gamma(z)^k f(z), \quad \text{for all } \gamma \in \Gamma,$$

where $j_\gamma(z) = \frac{cz+d}{|cz+d|}$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We equip the space of weight k automorphic forms with the Peterson innerproduct

$$\langle f, g \rangle = \int_{X_\Gamma} f(z)\overline{g(z)} \frac{dxdy}{y^2}.$$

If f is L^2-integrable with respect to the Petersson innerproduct and an eigenfunction of the weight k Laplacian

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \frac{\partial}{\partial x},$$

then we say that f is a *Maaß form of weight k*. Finally, if f is rapidly decaying at the cusps of Γ we say that f is a *Maaß cusp form of weight k*. If a Maaß form of weight k is not cuspidal nor constant we call it a *residual Maaß form of weight k*. We will refer to Maaß forms of weight 0 simply as *Maaß forms*. Note that if $g \in \mathscr{S}_k(\Gamma)$ is a weight k holomorphic cusp form for Γ then $z \mapsto y^{k/2} g(z)$ defines a Maaß cusp form of weight k with eigenvalue $\frac{k}{2}(1-\frac{k}{2})$. We will refer to these automorphic forms simply as *holomorphic cusp forms*.

We will also consider certain non-square integrable automorphic forms, namely the Eisenstein series. We may assume that Γ has a cusp at ∞ of width one. Then we define the *Eisenstein series of weight k (at ∞)*:

$$E_k(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j_\gamma(z)^{-k}\,\mathrm{Im}(\gamma z)^s, \quad \mathrm{Re}\, s > 1,$$

and elsewhere by meromorphic continuation. We will be interested in the Eisenstein series with spectral parameter $\mathrm{Re}\, s = 1/2$ (which is exactly what shows up in the spectral theorem) and simply refer to an *Eisenstein series of weight k* as an automorphic form of the type $E_k(z, \frac{1}{2}+it)$ where $t\in\mathbb{R}$.

Let f be either a Maaß form or an Eisenstein series of weight k for Γ. Denote by F the lift of f to to $\mathbf{T}^1(X_\Gamma)$, given by

$$F(g) := j_g(i)^{-k} f(gi).$$

Then the geodesic period associated to f and $\mathscr{C}$ is given by

$$\mathscr{P}_f(\mathscr{C}) := \int_{\mathscr{C}} F(g)\,d\mu_{\mathscr{C}}(g). \tag{3.1}$$

If the closed geodesic $\mathscr{C}$ corresponds to the hyperbolic conjugacy class $\{\gamma\}$, we may write $\mathscr{P}_f(\{\gamma\}) = \mathscr{P}_f(\mathscr{C})$.

3.2. **From geodesic to vertical periods.** We are now ready to state the main connection between vertical line integrals ins geodesic periods as was proved in [24, Prop. 4.2] in a slightly different form. The argument directly yields the following and we just indicate the changes needed.

Proposition 3.1. *Let $f:\mathbb{H}\to\mathbb{C}$ be a cuspidal Maaß form of weight k for Γ. Let $\gamma\in\Gamma$ be a hyperbolic matrix with lower left entry $c_\gamma>0$, and let $\mathscr{C}\subset\Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$ be the oriented closed geodesic corresponding to the Γ-conjugacy class of γ.*

Then we have for $\varepsilon>0$ that

$$\mathscr{P}_f(\mathscr{C}) = (-1)^{k/2+1}\int_0^\infty f(\gamma\infty + iy)\frac{dy}{y} + O_{f,\varepsilon}\left(1+\left(\frac{c_\gamma}{|\,\mathrm{tr}(\gamma)|}\right)^{1/2+\varepsilon}\right). \tag{3.2}$$

Similarly for f either an Eisenstein series or a residual Maaß form of weight k with Laplace eigenvalue $1/4+t^2$, we have

$$\begin{aligned}\mathscr{P}_f(\mathscr{C}) =&(-1)^{k/2+1}\int_0^\infty \left(f\left(\gamma\infty+iy\right)-f_\infty\left(y\right)\right)\frac{dy}{y}\\ &+A_f c_\gamma^{-1/2-it}\frac{1+(-1)^{k/2}}{1/2+it}+B_f c_\gamma^{-1/2+it}\frac{1+(-1)^{k/2}}{1/2-it}\\ &+O_{f,\varepsilon}\left(\left(\frac{c_\gamma}{|\operatorname{tr}(\gamma)|}\right)^{1/2+\varepsilon}+\left(\frac{c_\gamma}{|\operatorname{tr}(\gamma)|}\right)^{-1/2-\varepsilon}\right),\end{aligned}\tag{3.3}$$

where $f_\infty(y)=A_f y^{1/2-it}+B_f y^{1/2+it}$ denotes the constant Fourier coefficient at ∞.

Proof sketch. The formula in [24, Prop 4.2] gives a relation in the case $\Gamma=\Gamma_0(N)$ between the geodesic periods of an automorphic form f and additive twist L-series of f, which are certain linear combinations of vertical line integrals of f. On the one hand, we need to argue that the formulas (3.2) and (3.3) correspond exactly to the ones in [24, Prop 4.2] by rewriting the additive twists as line integrals. Secondly, we want a similar formula for general Γ. To obtain this, notice that the arguments in [24, Sec. 4] carry over to general Fuchsian groups of the first kind with a cusp at infinity of width one. The only difference being that the dependence on the spectral data of f might change, see Remark 4.1 in *loc. cit.*. Now the the claimed result follows by inserted equations (4.31) and (4.32) of *loc. cit.* into equation (4.27) of *loc. cit.* and doing the change of variables $z\mapsto\gamma z$ in the line integral in (4.32). ☐

3.3. **Normal distribution of vertical periods.** A key input in our proofs are normal distribution of vertical periods of automorphic forms as explored by many authors. These results were all motivated by the conjectures of Mazur–Rubin–Stein [20] originally formulated for weight 2 holomorphic forms (corresponding to modular symbols). These conjectures were motivated by connections to automorphic L-functions: Let f be an automorphic form for Γ, then for $\gamma\in\Gamma$ and $\operatorname{Re}s>1$ we have

$$\int_0^\infty (f(\gamma\infty+iy)-f_\infty(y))y^s\frac{dy}{y}=\gamma(s)\sum_{n\in\mathbb{Z}}a_f(n)e(n\gamma\infty)n^{-s}\tag{3.4}$$

$$=:\gamma(s)L(f,\gamma\infty,s),\tag{3.5}$$

where $a_f(n)$ denotes the Fourier coefficients of f (properly normalized) and $\gamma(s)$ is a certain special function. We refer to the Dirichlet series $L(f,\gamma\infty,s)$ as the *additive twist L-series of f*, which satisfies analytic continuation by the above formula satisfying a functional equation relating $L(f,\gamma\infty,s)$ and $L(f,\gamma^{-1}\infty,1-s)$ (for details we refer to [9, Sec. 5]). In the case where $\Gamma=\Gamma_0(N)$ and f is a Hecke eigenform, the additive twist L-series with denomitor q are dual to the twisted L-functions $L(f,\chi,s)=\sum_{n\geq1}\lambda_f(n)\chi(n)n^{-s}$ where χ is a Dirichlet character modulo q via the Birch–Stevens formula.

The following central limit theorem with effective rate of convergence has been obtained by, respectively, the first named author [7] using perturbation theory and by Sary Drappeau and the second named [9, Thm. 1.5] using dynamics of the Gauß map (we notice that the effective rate of convergence is not explicitly stated in *loc. cit.* but follows as in [2, p. 1412] using [9, Prop. 7.2] and the Berry–Esseen inequality).

Theorem 3.2 (C., Drappeau–N.)**.** *Let f be a Maaß cusp form or holomorphic cusp form for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, or f a weight 2 holomorphic form for a Fuchsian group Γ of the first kind with a cusp at ∞ of width one. Then for a rectangle $\mathscr{R} \subset \mathbb{C}$, we have*

$$\frac{1}{|X_\Gamma(N)|}\left|\left\{[\gamma]_\infty \in T_\Gamma(N) : \frac{\int_0^\infty f(\gamma\infty + iy)\frac{dy}{y}}{C_f\sqrt{\log N}} \in \mathscr{R}\right\}\right| = \frac{1}{2\pi}\int_{x+iy\in\mathscr{R}} e^{-\frac{x^2+y^2}{2}}dxdy + O_f\left(\frac{1}{\sqrt{\log N}}\right), \tag{3.6}$$

where $C_f > 0$. The implied constant may depend on f but is independent of $\mathscr{R}$.

For holomorphic forms of general weight k with respect to a general Fuchsian group of the first kind with a cusp, the normal distribution has been obtained by the second named author [23] using techniques from spectral theory. In this setting the rate of convergence is not known. The normal distribution of vertical periods of Eisenstein series was essentially achieved by Bettin–Drappeau [2, Thm. 2.1]. In this case an optimal rate of convergence is known of size $\frac{1}{(\log\log Q)^{1-\epsilon}}$.

Theorem 3.3 (N., Bettin–Drappeau)**.** *Let f be either a holomorphic cusp form for a Fuchsian group Γ of the first kind with a cusp or an Eisenstein series for $\mathrm{PSL}_2(\mathbb{Z})$. Then for a rectangle $\mathscr{R} \subset \mathbb{C}$, we have*

$$\frac{1}{|X_\Gamma(N)|}\left|\left\{[\gamma]_\infty \in X_\Gamma(N) : \frac{\int_0^\infty f(\gamma\infty + iy)\frac{dy}{y}}{C_f\sqrt{\log N}(\log\log N)^{\delta_f}} \in \mathscr{R}\right\}\right| \tag{3.7}$$

$$= \frac{1}{2\pi}\int_{x+iy\in\mathscr{R}} e^{-\frac{x^2+y^2}{2}}dxdy + o(1), \tag{3.8}$$

where $C_f > 0$, $\delta_f = 0$ for f cuspidal and $\delta_f = 3/2$ for f Eisenstein. The implied constant may depend on both f and $\mathscr{R}$.

4. Measures on bipartite graphs

In this section we will introduce an abstract framework for transferring measures from one component to another in a bipartite graph. We will in the next section apply this to graphs constructed from double cosets and conjugacy classes.

Let $(X,\mu),(Y,\nu)$ be discrete probability (measure) spaces, i.e. X, Y are finite sets equipped with the σ-algebra consisting of all subsets and μ (resp. ν) are probability measures on X (resp. Y). We define a *bipartite graph on* X, Y as a subset of the product

$$G \subset X \times Y.$$

We define *the neighbors of* $x \in X$ *(resp.* $y \in Y$*)* as

$$e(x) := \{y \in Y : (x, y) \in G\} \subset Y, \quad e(y) := \{x \in X : (x, y) \in G\} \subset X,$$

and for a subset $A \subset X$ (resp. $B \subset Y$) we denote

$$e(A) = \bigcup_{x \in A} e(x) \subset Y, \quad (\text{resp. } e(B) = \bigcup_{y \in B} e(y) \subset X).$$

We define *the degree of* x *(resp.* y*)* as

$$\deg(x) := \#e(x), \quad (\text{resp. } \deg(y) := \#e(y)).$$

We will denote by $G(\mu)$ the G*-transform of* μ which is the probability measure on Y given by

$$G(\mu)(\{y\}) := \sum_{x \in e(y)} \frac{\mu(x)}{\deg(x)}, \quad y \in Y, \tag{4.1}$$

and similarly we define the measure $G(\nu)$ on X. Note that for $x \in e(y)$ we have $y \in e(x)$ and thus $\deg(x) \geq 1$.

For a set $B \subset Y$ (resp. $A \subset X$), we denote

$$e^{-1}(B) := \{x \in X : e(x) \subset B\}, \quad (\text{resp. } e^{-1}(A) := \{y \in Y : e(y) \subset A\}).$$

Note that by definition $e^{-1}(B) \subseteq e(B)$.

Lemma 4.1. *For any subset* $B \subset Y$ *we have that*

$$\mu(e^{-1}(B)) \leq G(\mu)(B) \leq \mu(e(B)).$$

Proof. We have by definition of $G(\mu)$:

$$\begin{aligned} G(\mu)(B) = \sum_{y \in B} \sum_{x \in e(y)} \frac{\mu(x)}{\deg(x)} &= \sum_{x \in e(B)} \frac{\mu(x)}{\deg(x)} \cdot \#(e(x) \cap B) \\ &\leq \sum_{x \in e(B)} \frac{\mu(x)}{\deg(x)} \cdot \deg(x) = \mu(e(B)). \end{aligned}$$

Similarly,

$$G(\mu)(B) = \sum_{x \in e(B)} \frac{\mu(x)}{\deg(x)} \cdot \#(e(x) \cap B) \geq \sum_{x \in e^{-1}(B)} \frac{\mu(x)}{\deg(x)} \cdot \deg(x) = \mu(e^{-1}(B)).$$

$\square$

4.1. **Lifting the distribution.** Let $(G_n)_{n\geq 1}$ be a sequence of bipartite graphs on $(X_n, Y_n)_{n\geq 1}$. For each $n \geq 1$ let X_n be equipped with a probability measure μ_n. In addition, suppose we have $f_n : X_n \to \mathbb{R}$ such that $(f_n)^*(\mu_n)$ converges as $n \to \infty$ to a distribution on $\mathbb{R}$ with density function $F : \mathbb{R} \to \mathbb{R}_{\geq 0}$. In other words, for and $a < b$, we have

$$\lim_{n\to\infty} \mu_n\left(\{x \in X_n : f_n(x) \in [a, b]\}\right) = \int_a^b F(x)dx.$$

Moreover, suppose we know precise rate of convergence, i.e. there exists $h : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\lim_{x\to\infty} h(x) = 0$ and

$$\sup_{a<b} \left| \mu_n\left(\{x \in X_n : f_n(x) \in [a, b]\}\right) - \int_a^b F(x)dx \right| = O(h(n)). \tag{4.2}$$

We want to show that if we can define $g_n : Y_n \to \mathbb{R}$ such that $f_n(x)$ and $g_n(y)$ are "close" whenever x and y are connected by an edge in the bipartite graph G_n, then we can obtain information about the distribution of $(g_n)^*(G_n(\mu_n))$. In other words, using the structure of the graph G_n, we can "lift" the limit distribution of f_n on X_n to a limit distribution on Y_n.

Lemma 4.2. *Let $(G_n)_{n\geq 1}$ be a sequence of bipartite graphs on $(X_n, Y_n)_{n\geq 1}$. For each $n \geq 1$ let μ_n be a probability measure on X_n and let $f_n : X_n \to \mathbb{R}$ be as above such that (4.2) holds for a continuous density function F and some $h : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\lim_{x\to\infty} h(x) = 0$. Moreover, suppose there exists $g_n : Y_n \to \mathbb{R}$ such that whenever there is an edge between $x \in X_n$ and $y \in Y_n$ in G_n, we have $g_n(y) = f_n(x)+O(E(n))$, where $E : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\lim_{x\to\infty} E(x) = 0$. Then we have*

$$\sup_{a<b} \left| G_n(\mu_n)(\{y \in Y_n : g_n(x) \in [a, b]\}) - \int_a^b F(x)dx \right| = O_F(h(n) + E(n)),$$

where the implied constant is independent of $a < b$.

Proof. Fix $a < b$ and let

$$B_n := \{y \in Y_n : g_n(y) \in [a, b]\}.$$

Using Lemma 4.1, we have that

$$\mu_n(e^{-1}(B_n)) \leq G_n(\mu)(B_n) \leq \mu_n(e(B_n)).$$

Moreover, we have that

$$e(B_n) \subseteq \{x \in X_n : f_n(x) \in [a - E(n), b + E(n)]\}$$

and hence

$$G_n(\mu_n)(B_n) \leq \int_{a-E(n)}^{b+E(n)} F(x)dx + O(h(n)) = \int_a^b F(x)dx + O_F(h(n) + E(n)).$$

Here we used that F is a continuous density function, therefore indeed we have

$$\int_b^{b+E(n)} F(x)dx = O_F(E(n)).$$

Similarly, we have that

$$e^{-1}(B_n) \supseteq \{x \in X_n : f_n(x) \in [a + E(n), b - E(n)]\}$$

and the conclusion follows. □

Corollary 4.3. *In the same setting as above. Then for any function $H : \mathbb{N} \to \mathbb{R}_{>0}$ such that $H(n) \to \infty$ as $n \to \infty$ it holds that*

$$G_n(\mu_n) \left(\{y \in Y_n : H(n)^{-1} \le |g_n(x)| \le H(n)\}\right) \to 1, \quad n \to \infty$$

Proof. This follows since the density function F is assumed continuous. □

It is natural to ask a similar question about complex-valued functions $f_n : X_n \to \mathbb{C}$ and $g_n : Y_n \to \mathbb{C}$. In this case, we can obtain similar result about distributions $(f_n)^*(\mu_n)$ and $(g_n)^*(G(\mu_n))$, but without uniformity.

Suppose we know that there exists a density function $F : \mathbb{C} \to \mathbb{R}_{\ge 0}$ such that for all rectangles

$$\mathscr{R} = \mathscr{R}(w; R_1, R_2) := \{z \in \mathbb{C} : -R_1 \le \mathrm{Re}(z - w) \le R_1, -R_2 \le \mathrm{Im}(z - w) \le R_2\},$$

we have that

$$\lim_{n\to\infty} \mu_n \{x \in X_n : f_n(x) \in \mathscr{R}\} = \int_{\mathscr{R}} F(z)dz. \tag{4.3}$$

Lemma 4.4. *Let $f_n : X_n \to \mathbb{C}$ such that $(f_n)^*(\mu_n)$ obeys asymptotically a distribution given by continuous density function F, that is (4.3) holds. Let $g_n : Y_n \to \mathbb{C}$ such that whenever $x \in X_n$ and $y \in Y_n$ are connected by an edge in G_n, we have $|f_n(x) - g_n(y)| \le E(n)$ for some $E : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ with $\lim_{n\to\infty} E(n) = 0$. Then for any rectangle $\mathscr{R} = \mathscr{R}(w; R_1, R_2) \subset \mathbb{C}$, we have*

$$\lim_{n\to\infty} G_n(\mu_n) \left(\{y \in X_n : g_n(y) \in \mathscr{R}\}\right) = \int_{\mathscr{R}} F(z)dz. \tag{4.4}$$

Proof. We proceed as in the proof of Lemma 4.2. Let

$$A_n := \{y \in Y_n : g_n(y) \in \mathscr{R}\}.$$

Then

$$e(A_n) \subseteq \{x \in X_n : f_n(x) \in \mathscr{R}(w; R_1 + E(n), R_2 + E(n))\},$$

by the assumption on f_n and g_n. Similarly,

$$e(A_n) \supseteq \{x \in X_n : f_n(x) \in \mathscr{R}(w; R_1 - E(n), R_2 - E(n))\}.$$

Using Lemma 4.1, we have

$$\begin{aligned}&\mu_n(\{f_n^{-1}(\mathscr{R}(w;R_1-E(n),R_2-E(n))\})\\&\leq G_n(\mu_n)(A_n)\leq\mu_n(\{f_n^{-1}(\mathscr{R}(w;R_1+E(n),R_2+E(n))\}).\end{aligned}$$

Conclusion follows by letting $n\to\infty$ since the density function F is assumed continuous. □

Remark 4.5. Using this approach, in the complex case we cannot obtain uniformity in the error term, for all rectangles $\mathscr{R}\subset\mathbb{C}$. This boils down to the fact that the error term is given by

$$\int_{\mathscr{R}(w;R_1+E(n),R_2+E(n))}F(z)dz-\int_{\mathscr{R}(w;R_1,R_2)}F(z)$$

and the region $\mathscr{R}(w;R_1+E(n),R_2+E(n))\backslash\mathscr{R}(w;R_1,R_2)$ has size $\approx 2(R_1+R_2)\cdot E(n)$ (as opposed to size $E(n)$ in the one dimensional real case).

5. Bipartite graphs from double cosets and conjugacy classes

Let $\Gamma\subset\mathrm{PSL}_2(\mathbb{R})$ be a discrete and cofinite subgroup with a cusp at infinity of width 1 (i.e. a non-cocompact Fuchsian group of the first kind). Let

$$\mathrm{Conj}(\Gamma):=\{\{\gamma\}:\gamma\in\Gamma\},\quad\{\gamma\}:=\{\sigma\gamma\sigma^{-1}:\sigma\in\Gamma\}\subset\Gamma,$$

be the set of (not necessarily primitive) conjugacy classes of Γ. Denote by $\Gamma_\infty=\{\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right):n\in\mathbb{Z}\}$ the stabilizer of the cusp ∞. Let

$$\Gamma_\infty\backslash\Gamma/\Gamma_\infty:=\{[\gamma]_\infty:\gamma\in\Gamma\},\quad[\gamma]_\infty:=\Gamma_\infty\gamma\Gamma_\infty=\{\sigma_1\gamma\sigma_2:\sigma_1,\sigma_2\in\Gamma_\infty\}\subset\Gamma,$$

be the set of double cosets with respect to the pair of cusps (∞,∞). We will apply the general frame work developed in the previous section to the sets:

$$X_N:=\{[\gamma]_\infty\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty:0<|c_\gamma|\leq N\},\tag{5.1}$$

$$Y_N:=\{\{\gamma\}\in\mathrm{Conj}(\Gamma):2<|\operatorname{tr}(\gamma)|\leq N\},\tag{5.2}$$

for $N\geq 1$. Note that by, respectively, [28, Eq. (3.6)] and the prime geodesic theorem [15] we have

$$|X_N|\sim C_1N^2,\quad|Y_N|\sim C_2\tfrac{N^2}{\log N},\quad N\to\infty\tag{5.3}$$

for certain constants $C_1,C_2>0$ depending on Γ. We denote $Y_N^*\subset Y_N$ the subset of *primitive* conjugacy classes. Recall also that the non-primitive conjugacy classes are negligible

$$|Y_N\setminus Y_N^*|\ll N.\tag{5.4}$$

We define the bipartite graph

$$G_N:=\{([\gamma]_\infty,\{\gamma'\})\in X_N\times Y_N:[\gamma]_\infty\cap\{\gamma'\}\neq\emptyset\}.$$

Given an element $x=[\gamma]_\infty\in X_N$ we define

$$c(x):=|c_\gamma|,$$

to be the absolute lower-left entry of a representative γ of the double coset (which we note is independent of the choice of representative). Note that when $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ then $c(x) = \mathrm{denom}(\gamma\infty)$ is the denominator of the rational number $\gamma\infty \in \mathbb{Q}$.

For $y = \{\gamma\} \in Y_N$ we define $\mathscr{C}_y \subset X_\Gamma$ to be the oriented, closed geodesic corresponding to the (hyperbolic) conjugacy class $\{\gamma\}$ and put

$$\ell(y) := \ell(\mathscr{C}_y).$$

We equip X_N and Y_N with the probability measures μ_N and ν_N proportional to the counting measures:

$$\mu_N(\{x\}) = \frac{1}{|X_N|}, \quad \nu_N(\{y\}) = \frac{1}{|Y_N|}.$$

Let f be a Maaß form, a holomorphic form or an Eisenstein series of weight k for Γ. For $x = [\gamma]_\infty \in X_N$, denote

$$L_f(x) := \int_0^\infty f(\gamma\infty + iy)\frac{dy}{y}.$$

For $y \in Y_N$, denote the associated geodesic period by

$$\mathscr{P}_f(y) := \mathscr{P}_f(\mathscr{C}_y),$$

where $\mathscr{P}_f(\mathscr{C}_y)$ is defined as in equation (3.1). We will use the tools developed in Section 4 to transfer information about the distribution of the values $\{L_f(x) : x \in X_n\}$ to those of $\{\mathscr{P}_f(y) : y \in Y_n\}$.

5.1. **Normal distribution.** In this section we will "lift" the normal distribution of the vertical periods $L_f(x), x \in X_N$ to the geodesic periods using the result of Section 4.1. For this purpose it will be convenient to define

$$Y'_N = \{\{\gamma\} \in \mathrm{Conj}(\Gamma) : N/2 \leq |\operatorname{tr}(\gamma)| \leq N\},$$

and work with the restricted graph

$$G'_N = \{([\gamma]_\infty, \{\gamma'\}) \in X_N \times Y'_N : [\gamma]_\infty \cap \{\gamma'\} \neq \emptyset\}.$$

Working with this graph will allow us to control the size of error terms.

To simplify notation, we denote by $\mu'_N : Y'_N \to \mathbb{R}_{\geq 0}$ the G'_N-transform of the probability measure μ_N on X_N proportional to the counting measure, that is

$$\mu'_N(\{y\}) := G'_N(\mu_N)(\{y\}) = \frac{1}{|X_N|}\sum_{x \in e(y)} \frac{1}{\deg(x)}.$$

Corollary 5.1. *Let f be a Maaß cusp form or a holomorphic cusp form for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, or let f be a weight 2 holomorphic form for a general Fuchsian group Γ of the first kind with a cusp at infinity of width one. Then for any $a < b$, we have*

$$\mu'_N\left(\left\{y \in Y'_N : \mathrm{Re}\left(\frac{\mathscr{P}_f(y)}{C_f\sqrt{\log N}}\right) \in [a,b]\right\}\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{x^2}{2}}\,dx + O\left(\frac{1}{\sqrt{\log N}}\right).$$

Proof. Combine Proposition 3.1, Theorem 3.2, and Lemma 4.2 applied to the graph G'_N and

$$f_N(x) = \mathrm{Re}\left(\frac{L_f(x)}{C_f\sqrt{\log N}}\right), \quad g_N(y) = (-1)^{k/2+1}\,\mathrm{Re}\left(\frac{\mathscr{P}_f(y)}{C_f\sqrt{\log N}}\right).$$

□

Corollary 5.2. *Let f be a Maaß cusp form or holomorphic cusp form for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, or f a holomorphic cusp form for a general fuchsian group Γ with a cusp. Then for any rectangle $\mathscr{R} \subset \mathbb{C}$, we have*

$$\lim_{N\to\infty} \mu'_N\left(\left\{y \in Y'_N : \frac{\mathscr{P}_f(y)}{C_f\sqrt{\log N}} \in \mathscr{R}\right\}\right) = \frac{1}{2\pi}\int_{x+iy\in\mathscr{R}} e^{-\frac{x^2+y^2}{2}}\,dxdy.$$

Proof. Combine Proposition 3.1, Theorem 3.3, and Lemma 4.4 applied to the graph G'_N and

$$f_N(x) = \mathrm{Re}\left(\frac{L_f(x)}{C_f\sqrt{\log N}}\right), \quad g_N(y) = (-1)^{k/2+1}\,\mathrm{Re}\left(\frac{\mathscr{P}_f(y)}{C_f\sqrt{\log N}}\right).$$

□

The last two corollaries show that if we count closed geodesics with “weights” inferred by the graph G'_N, then we obtain normal distribution of the geodesics periods. As mentioned in the introduction we believe that the normal distribution should hold for the counting measure on Y'_N. For the rest of this paper, by studying properties of this graph, we obtain results about the distribution of the geodesic periods with respect to the counting measure ν_N, in particular with applications to the zero set.

5.2. **Controlling the degrees of the graphs G_N.** Recall the definitions (5.1) and (5.2) of the discrete spaces (X_N, μ_N) and (Y_N, ν_N) where $N \geq 1$, defined from, respectively, double cosets and conjugacy classes of a Fuchsian group Γ of the first kind with a cusp at infinity of width one.

As mentioned in the introduction, the proof of Theorem 1.4 amounts to an upper bound on the degrees $\deg(x)$ for $x \in X_N$ and a lower bound for the degrees $\deg(y)$ for $y \in Y_N$ (on average).

5.2.1. *Degrees of vertices in X_N.* This first bound is easily controlled as follows.

Lemma 5.3. *For $x \in X_N$ it holds that*

$$\deg(x) = \frac{2N+1}{c(x)} + E(x),$$

where $|E(x)| \leq 1 + \lfloor \frac{5}{c(x)} \rfloor$. In particular, for $N \geq 5$ we have $\deg(x) \geq 1$ for all $x \in X_N$.

Proof. Let $x \in X_N$ and let $\gamma \in \Gamma$ be such that $x = [\gamma]_\infty$ and $c(x) = c_\gamma > 0$. Since multiplication from the left by the matrix $T^{\pm 1}$ changes the trace by $\pm c_\gamma$, we may arrange it so that $-c(x)/2 \le a_\gamma + d_\gamma < c(x)/2$. We note that for any $k \in \mathbb{Z} \setminus \{0\}$ the conjugacy classes of γ and $T^k\gamma$ are different as the (signed) traces are different. This implies that

$$\deg(x) = \#\{k \in \mathbb{Z} : 2 < |a_\gamma + d_\gamma + kc(x)| \le N\} \tag{5.5}$$

$$= \#\{k \in \mathbb{Z} : |a_\gamma + d_\gamma + kc(x)| \le N\} - \#\{k \in \mathbb{Z} : |a_\gamma + d_\gamma + kc(x)| \le 2\}. \tag{5.6}$$

This yields the wanted equality since

$$\lfloor \tfrac{2N+1}{c(x)} \rfloor \le \#\{k \in \mathbb{Z} : |a_\gamma + d_\gamma + kc(x)| \le N\} \le \lceil \tfrac{2N+1}{c(x)} \rceil,$$

and

$$\#\{k \in \mathbb{Z} : |a_\gamma + d_\gamma + kc(x)| \le 2\} \le \lfloor \tfrac{5}{c(x)} \rfloor.$$

□

5.2.2. *The degrees of vertices in Y_N.* We will now give a geometric lower bound for the degrees $\deg(y)$ with $y \in Y_N$. Let $T_\Gamma \ge 1$ be such that

$$\mathscr{B} := \{z \in \mathbb{H} : -1/2 < \operatorname{Re} z \le 1/2, T_\Gamma \le \operatorname{Im} z \le 2T_\Gamma\} \subset X_\Gamma.$$

This exists by the thick-thin decomposition (see e.g. [39, Chap. 4.5]). We have the following key geometric lower bound for the degrees of vertices in Y_N.

Proposition 5.4. *Let $N \ge 1$ and let $\mathscr{B}$ be as above. Then there exists a constant $C = C(\mathscr{B}) > 0$ (independent of N) such that for any $y \in Y_N^*$, we have*

$$\deg(y) \ge C \cdot \ell(\mathscr{C}_y \cap \mathscr{B}).$$

We start by recalling the following hyperbolic geometric fact.

Lemma 5.5. *Let $0 < T \le r$ and let S denote the infinite geodesic connecting $-r$ and r. Then $\ell(S \cap \{\operatorname{Im} z \ge T\}) = 2\log\Big(r + \sqrt{r^2 - T^2}\Big) - 2\log T$.*

Proof. Applying the matrix

$$\begin{pmatrix} (2r)^{-1/2} & -(r/2)^{1/2} \\ (2r)^{-1/2} & (r/2)^{1/2} \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}),$$

takes S to the vertical geodesic from 0 to ∞. The two points in $S \cap \{\operatorname{Im} z = T\}$ are taken to, respectively $z_0 = i\frac{T^2}{r - \sqrt{r^2 - T^2}}$ and $z_1 = i\frac{T^2}{r + \sqrt{r^2 - T^2}}$. Now the first result follows since Möbius transformations preserve the geodesic length and

$$\int_{z_0}^{z_1} \frac{|dz|}{y} = \log\left(r + \sqrt{r^2 - T^2}\right) - \log\left(r - \sqrt{r^2 - T^2}\right) = 2\log\left(r + \sqrt{r^2 - T^2}\right) - 2\log T,$$

as wanted. □

Corollary 5.6. *Let $T>0$ and $r>0$ and let S denote the infinite geodesic connection $-r$ and r. Then it holds that*

$$\ell(S\cap\{T\le \operatorname{Im} z\le 2T\})\le 2\log\Big(2+\sqrt{3}\Big).$$

Proof. If $r<T$ then clearly the intersection is empty. Assume that $T\le r<2T$ then by the above lemma we get that

$$\ell(S\cap\{\operatorname{Im} z\ge T\})=2\log\Big(r+\sqrt{r^2-T^2}\Big)-\log T\le 2(2+\sqrt{3}).$$

Assume finally that $r>2T$. Then we have by the above that

$$\ell(S\cap\{\operatorname{Im} z\ge T\})=2\log\Big(r+\sqrt{r^2-T^2}\Big)-2\log T-2\log\Big(r+\sqrt{r^2-4T^2}\Big)+2\log 2T,$$

which has r-derivative equal to

$$\frac{\sqrt{r^2-4T^2}-\sqrt{r^2-T^2}}{\sqrt{r^2-4T^2}\sqrt{r^2-T^2}}.$$

Thus it is a decreasing function for $r\ge 2T$ and at $r=2T$ we recover the same value as in the previous case. □

Proof of Proposition 5.4. For $\gamma\in\Gamma$ hyperbolic and primitive, denote by S_γ the axis of γ, i.e. the infinite geodesic half-circle with end-points in $\mathbb{R}$ given by

$$\frac{a-d\pm\sqrt{(a+d)^2-4}}{2c}.$$

For $\gamma,\gamma'\in\Gamma$ primitive and hyperbolic one has $S_\gamma=\delta S_{\gamma'}\iff\gamma'=\delta\gamma^{\pm1}\delta^{-1}$. Also, note that

$$\ell(\Gamma_\infty S_\gamma\cap\mathscr{B})=\ell(S_\gamma\cap\Gamma_\infty\mathscr{B})=\ell(S_\gamma\cap\{T_\Gamma\le\operatorname{Im} z\le 2T_\Gamma\})\ll_\Gamma 1,\tag{5.7}$$

where we used Corollary 5.6. Put in words, this means that the length of all translations of S_γ intersected with $\mathscr{B}$ is bounded above by a constant.

Now fix a primitive closed geodesic $\mathscr{C}$ and consider its projection on X_Γ. We consider the geodesic arcs of $\mathscr{C}$ that intersect $\mathscr{B}$. We have that

$$\mathscr{C}\cap\mathscr{B}=\bigcup_{\gamma\in y(\mathscr{C})}(S_\gamma\cap\mathscr{B}),$$

where $y(\mathscr{C})\in\operatorname{Conj}(\Gamma)$ denotes the primitive conjugacy class associated to the closed geodesic $\mathscr{C}$. Note that if $S_\gamma\cap\mathscr{B}\ne\emptyset$, then clearly the radius of axis of γ satisfies $r_\gamma\ge T_\Gamma\ge 1$. By the explicit formula:

$$r_\gamma=\frac{\sqrt{\operatorname{tr}(\gamma)^2-4}}{2c_\gamma},$$

we conclude that $c_\gamma \leq |\operatorname{tr}(\gamma)|$. This shows that if $\{\gamma\} \in Y_N^*$ and $S_\gamma \cap \mathscr{B} \neq \emptyset$ then we have $[\gamma]_\infty \in X_N$, and thus $e = ([\gamma]_\infty, \{\gamma\})$ is an edge of G_N. Now if $\gamma' \in \Gamma$ satisfies $\gamma' \in \{\gamma\}$ and $\gamma' \in [\gamma]_\infty$ then we have

$$\operatorname{tr}(\gamma) = \operatorname{tr}(\gamma') \quad \text{and} \quad \gamma = T^m \gamma' T^n,$$

for some $m, n \in \mathbb{Z}$. Observe that we have $\operatorname{tr}(T^m\gamma'T^n) = \operatorname{tr}(\gamma') + (m-n)c_{\gamma'}$ where $c_{\gamma'} \neq 0$ denotes the lower-left entry of γ' (which is non-zero since γ' is hyperbolic). This means that $m = n$ and thus the elements $\gamma' \in y(\mathscr{C})$ contributing to the edge e are exactly the conjugates by Γ_∞ of γ. But from (5.7), we know that the total contribution of these conjugates to the total length of $\mathscr{C} \cap \mathscr{B}$ is $O_\Gamma(1)$. This shows indeed that $\ell(\mathscr{C} \cap \mathscr{B}) \ll \deg(\{\gamma\})$ as wanted. □

5.2.3. *Sparse equidistribution of closed geodesics.* In this section we will prove a sparse equidistribution result using a technique of Aka–Einsiedler [1] which combined with Proposition 5.4 will allow us to control the degrees of Y_N.

For $T \geq 1$ we denote by

$$\phi_T(g) := \frac{1}{T}\int_0^T \phi(ga_t)dt, \quad a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \tag{5.8}$$

the average of ϕ under the geodesic flow. Then we have the following control of this action.

Lemma 5.7. *Let $\phi : \Gamma\backslash \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ be a compactly supported positive valued function. Let $F : \Gamma\backslash\mathrm{PSL}_2(\mathbb{R}) \to \mathbb{C}$ be an (L^2-normalized) Maaß form or an Eisenstein series of weight k and spectral parameter t_F. Then for each $A \geq 0$ there exists $B > 0$ such that it holds that*

$$\langle \phi_T, F\rangle \ll_{\phi,A} (1 + k^2 + |t_f|)^{-A} e^{BT} \quad T \to \infty,$$

where $\langle \cdot, \cdot \rangle$ denotes the Petersson innerproduct on $\Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$.

Proof. Consider the self-adjoint differential operator $\Delta' := 1 + \Delta + \frac{\partial^2}{\partial\theta^2}$ acting on (a dense subspace of) $L^2(\Gamma\backslash\mathrm{PSL}_2(\mathbb{R}))$. Then we have by self-adjointness that for any $n \geq 0$

$$\begin{aligned}\langle \phi_T, F\rangle &= (1 + k^2 + 1/4 + t_f^2)^{-n}\langle \phi_T, (\Delta')^n F\rangle \\ &\ll_n (1 + |k| + |t_f|)^{-2n} \|y_\Gamma^{1/2}(\Delta')^n\phi_T\| \cdot \|y_\Gamma^{-1/2}F\|,\end{aligned}$$

where $\|F\|^2 = \langle F, F\rangle$ in terms of the Petersson inner product and $y_\Gamma : X_\Gamma \to \mathbb{R}_{\geq 0}$ denotes the invariant height of Γ defined as in [17, (2.42)] (extended to a function on $\Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$ by $\mathrm{PSO}_2(\mathbb{R})$-invariance). Notice that the value of y_Γ on the support of ϕ_T is bounded by $O_\phi(e^T)$ and so $\|y_\Gamma^{1/2}(\Delta')^n\phi_T\|$ is bounded by $e^{T/2} \cdot S_{\infty,2n}(\phi_T)$ where $S_{\infty,2n}$ denotes the Sobolev norm from [40, Sec. 2.9.2]. It follows by Lemma 2.2 in *loc. cit.* that we have

$$|S_{\infty,2n}(\phi_T)| \ll_n e^{2nT} S_{\infty,2n}(\phi).$$

Finally, by standard sup norm bounds (this is proved in [17, Prop. 7.2] for weight 0 and a similar argument using a weight k automorphic kernel yields the general result) we know that $||y_\Gamma^{-1/2}F||$ is bounded polynomially in k and t_F for any F as above (including the case where F is an Eisenstein series due to the factor $y_\Gamma^{-1/2}$). This yields the wanted since we can pick n arbitrarily large. $\square$

Theorem 5.8. *Let Γ be a Fuchsian group of the first kind. Fix $\epsilon > 0$. For each $N \geq 1$ let*

$$I_N \subset C_\Gamma(N) = \{\mathscr{C} \subset X_\Gamma : \textit{closed geodesics of length} \leq N\}$$

be a subcollection of closed geodesics such that

$$\sum_{\mathscr{C}\in I_N} \ell(\mathscr{C}) \geq \epsilon \cdot \ell(C_\Gamma(N)) = \epsilon \cdot \sum_{\mathscr{C}:\ell(\mathscr{C})\leq N} \ell(\mathscr{C}).$$

Then the collections of closed geodesics I_N equidistribute in the unit tangent bundle of X_Γ as $N \to \infty$.

Proof. Let $f : \mathbb{H} \to \mathbb{C}$ be either a cuspidal or residual Maaß form or an Eisenstein series of weight k automorphic form for Γ. An equidistribution theorem of Zelditch [41, Thm 6.1] yields

$$\left| \frac{1}{\ell(C_\Gamma(N))} \sum_{\mathscr{C}\in C_\Gamma(N)} \mathscr{P}_f(\mathscr{C}) \right| \leq E_1(t_f, k)N^{-\delta}, \tag{5.9}$$

where $\delta > 0$ only depends on (the spectral gap of) Γ and $E_1 : \mathbb{C} \times \mathbb{Z} \to \mathbb{R}_{\geq 0}$ is some continuous (error-term) function depending on the spectral parameter t_f and the weight k of f. For the fact that the error-term depends continuously on the spectral data of F see the discussion on page 85 in *loc. cit.*. Recall the definition (5.8) of the average ϕ_T. Following Aka–Einsiedler [1] we have by invariance of $\mu_{\mathscr{C}}$ under the geodesic flow that for any $T \geq 1$ it holds that

$$\begin{aligned} \left| \frac{1}{\ell(I_N)} \sum_{\mathscr{C}\in I_N} \int_{\mathscr{C}} \phi \, d\mu_{\mathscr{C}} \right|^2 = \left| \frac{1}{\ell(I_N)} \sum_{\mathscr{C}\in I_N} \int_{\mathscr{C}} \phi_T \, d\mu_{\mathscr{C}} \right|^2 &\leq \frac{1}{\ell(I_N)} \sum_{\mathscr{C}\in I_N} \int_{\mathscr{C}} |\phi_T|^2 \, d\mu_{\mathscr{C}} \\ &\leq \frac{\epsilon^{-1}}{\ell(C_\Gamma(N))} \sum_{\mathscr{C}\in C_\Gamma(N)} \int_{\mathscr{C}} |\phi_T|^2 \, d\mu_{\mathscr{C}}. \end{aligned}$$

We want to show that we can pick $T \geq 1$ (depending on N) such that the right-hand side tends to zero as $N \to \infty$. We start by spectraly expanding:

$$|\phi_T|^2 = \langle 1, |\phi_T|^2 \rangle + \varphi_T^0 + \varphi_T^r + \varphi_T^c,$$

where

$$\varphi_T^0 = \sum_{k\in 2\mathbb{Z}} \sum_{F\in \mathscr{B}_k^0} \langle |\phi_T|^2, F\rangle F, \quad \varphi_T^r = \sum_{k\in 2\mathbb{Z}} \sum_{F\in \mathscr{B}_k^r} \langle |\phi_T|^2, F\rangle F, \tag{5.10}$$

$$\varphi_T^c = \sum_{k\in 2\mathbb{Z}} \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{\mathbb{R}} \langle |\phi_T|^2, E_{\mathfrak{a},k,it}\rangle E_{\mathfrak{a},k,it} dt \tag{5.11}$$

denotes, respectively, the cuspidal, (non-constant) residual and continuous projections of $|\phi_T|^2$. Here $\mathscr{B}_k^0$ and $\mathscr{B}_k^r$ denotes an orthonormal basis of, respectively, cuspidal Maaß forms and residual Maaß forms of weight k, and $\mathfrak{a}$ runs over a full set of inequivalent cusps of Γ and $E_{\mathfrak{a},k,it}$ denotes the corresponding Eisenstein series of weight k [11, Eq. (4.44)]. By Lemma 5.7 (and Weyl's law), we see that we can truncate each of the expansions for $|k| + |t_F| \le e^{T^2} N$, say, at the cost of an error term $O(e^{-T^2} N^{-100})$. By effective mixing as in [1, Prop. 4] we can bound the constant contribution by $\langle 1, |\phi_T|^2\rangle \ll_\phi \frac{1}{T}$. Finally, we apply Zelditch's result (5.9) to the remaining Maaß forms (i.e. those for which $|k| + |t_F| \le e^{T^2}$), and we conclude that

$$\sum_{\mathscr{C}\in C_\Gamma(N)} \int_{\mathscr{C}} |\phi_T|^2 \, d\mu_{\mathscr{C}} \ll_\phi \ell(C_\Gamma(N)) \left(\frac{1}{T} + E_2(T) N^{-\delta}\right),$$

for some continuous function $E_2 : \mathbb{R}_{\ge 1} \to \mathbb{R}_{\ge 0}$ depending on ϕ. The result now follows by picking

$$T = T(N) := \max\{t \ge 1 : E_2(t) \le N^\delta / \log N\},$$

and observing that $T(N) \to \infty$ as $N \to \infty$ by the continuity of the error-term E_2. □

In the case of the modular group we have the following slight improvement by a result of Aka–Einsiedler [1, Thm. 2].

Theorem 5.9. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ be the modular group. For each $N \ge 1$ let*

$$I_N \subset C(N) = \{\mathscr{C} \subset \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H} : \textit{closed geodesics of length} \le N\}$$

be a subcollection of closed geodesics such that

$$\sum_{\mathscr{C}\in I_N} \ell(\mathscr{C}) \gg \frac{\ell(C(N))}{\log N}.$$

Then the collections of closed geodesics I_N equidistribute in the unit tangent bundle of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ as $N \to \infty$.

Proof. As explained in [32] the primitive closed geodesics on the modular curve $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ are parameterized by elements of class groups Cl_D of quadratic orders of discriminant D. By a theorem of Aka–Einsiedler [1, Thm. 2] we know that for a subcollection $I_D \subset \mathrm{Cl}_D$ such that $|I_D| \ge \frac{|\mathrm{Cl}_D|}{\log D}$ then the associated geodesics equidistribute as $D \to \infty$. Since the length of the geodesics associated to an element of Cl_D is $\log \epsilon_D$ and $|\mathrm{Cl}_D| \log \epsilon_D = D^{1/2+o(1)}$ and $\ell(C(N)) = N^{2+o(1)}$, we get the wanted conclusion. □

Remark 5.10. It is likely that one can obtain a proof of the sparse equidistribution in Theorem 5.9 for a general Fuchsian group of the first kind. By the method of Aka–Einsiedler this reduces to proving a version of Zelditch's theorem (5.9) with a polynomial dependence on the spectral parameter of the Maaß form F. This should follow by a detailed analysis of the hypergeometric functions appearing in Zelditch's trace formula, see [41, p. 85], as well as invoking a version of Sarnak's bound for triple periods [33] with a polynomial dependence of the spectral datum of the "fixed" Maaß forms (as was carried out in a special case in [16, Appendix A]). We have not pursued this.

6. Non-vanishing of geodesic periods

In this section we will prove non-vanishing results for geodesic periods using the results in the previous sections. We will prove two versions: one where we obtain strong quantitative bounds on the number of "very small" geodesic periods, and on the other hand, we show that 100% of geodesic periods are not "too small".

The following is a slight generalization of Theorem 1.2.

Theorem 6.1. *Let f be a Maaß cusp form for the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and let $\delta > 0$. Then we have that*

$$\left|\left\{ y \in Y_N^* : |\mathscr{P}_f(y)| \leq (\log N)^{1/2-\delta} \right\}\right| \ll \frac{N^2}{(\log N)^{1+\min(\delta,1/4)}},$$

as $N \to \infty$.

Proof. Put $\alpha = 1 + \min(\delta, 1/4)$ and

$$A_N := \left\{ y \in Y_N^* : |\mathscr{P}_f(y)| \leq (\log N)^{1/2-\delta} \right\}.$$

Clearly we may suppose that $|A_N| \geq C \frac{N^2}{(\log N)^\alpha}$, where C is a sufficiently large constant.

Let

$$B_N := \left\{ y \in Y_N^* : \mathrm{tr}(y) \leq \frac{N}{(\log N)^{1/3}} \right\}.$$

Then by the prime geodesic theorem we have that

$$|B_N| \ll \left(\frac{N}{(\log N)^{1/3}}\right)^2 \left(\log\left(\frac{N}{(\log N)^{1/3}}\right)\right)^{-1} \ll \frac{N^2}{(\log N)^{5/3}}.$$

In particular, we see that by the assumption on the size of A_N we have

$$|A_N| \sim |A_N \setminus B_N|, \quad N \to \infty,$$

since $5/4 < 5/3$. This implies that as $N \to \infty$ we have

$$\sum_{y \in A_N \setminus B_N} \ell(\mathscr{C}_y) \gg \log N \frac{N^2}{(\log N)^\alpha} \gg \frac{N^2}{(\log N)^{1/4}} \tag{6.1}$$

by the definition of B_N.

Let $C_N := e(A_N \setminus B_N) \subset X_N$ be the set of neighbours of $A_N \setminus B_N$. From Proposition 3.1, we see that for $x \in C_N$

$$|L_f(x)| \leq |\mathscr{P}_f(y)| + O\left(1 + \left(\frac{N}{N/(\log N)^{1/3}}\right)^{1/2+\epsilon}\right) \ll (\log N)^{1/2-\delta} + (\log N)^{1/4},$$

for $0 < \epsilon < 1/100$, say. In particular, there exists some constant $C > 0$ such that $x \in C_N$ implies that $|L_f(x)/(\log N)^{1/2}| \leq C(\log N)^{-(\alpha-1)}$. From the normal distribution of the vertical periods as in Theorem 3.2, we have that

$$\frac{|C_N|}{|X_N|} \leq \int_{|z| \leq C(\log N)^{-(\alpha-1)}} \frac{e^{-|z|^2/2}}{2\pi} dz + O\left(\frac{1}{\sqrt{\log N}}\right) \ll (\log N)^{-2(\alpha-1)}, \tag{6.2}$$

using that $2(\alpha - 1) = 2\min(1/4, \delta) \leq 1/2$.

We now proceed to count the total number of edges out of both $A_N \setminus B_N$ and C_N in two different ways. On the one hand, we have

$$\sum_{y \in A_N \setminus B_N} \deg y \gg \sum_{y \in A_N \setminus B_N} l(\mathscr{C}_y \cap \mathscr{B}) \gg \sum_{y \in A_N \setminus B_N} l(\mathscr{C}_y) \gg |A_N \setminus B_N| \log N$$

where in the first inequality we applied Proposition 5.4 (recall the definition of $\mathscr{B}$), in the second we applied the equidistribution result Theorem 5.8 to the geodesics in $A_N \setminus B_N$ (which applies by the lower bound (6.1)), and in the last we used that for $y \in A_N \setminus B_N$ we have $\ell(\mathscr{C}_y) \gg \log N$ (by the definition of B_N).

On the other hand, we can upper bound the degrees in C_N as follows:

$$\begin{aligned}
\sum_{x \in C_N} \deg x = \sum_{\substack{x \in C_N: \\ c_x \leq \frac{N}{(\log N)^\beta}}} \deg x + \sum_{\substack{x \in C_N: \\ c_x > \frac{N}{(\log N)^\beta}}} \deg x &\leq \sum_{\substack{x \in X_N: \\ c_x \leq \frac{N}{(\log N)^\beta}}} \deg x + 2\sum_{x \in C_N} (\log N)^\beta \\
&\ll \frac{N^2}{(\log N)^\beta} + |C_N|(\log N)^\beta \\
&\ll \frac{N^2}{(\log N)^\beta} + \frac{N^2}{(\log N)^{2-2\alpha-\beta}},
\end{aligned}$$

where in the second to last inequality we have applied Lemma 5.3 and the last inequality follows from the upper bound (6.2) on the size of C_N. Choosing $\beta = \alpha - 1$, we obtain an upper bound of $N^2/(\log N)^{\alpha-1}$. Notice that since $C_N \subset X_N$ consists of all the neighbours of $A_N \setminus B_N$, we have the trivial inequality

$$\sum_{x\in C_N} \deg x \geq \sum_{y\in A_N\setminus B_N} \deg y$$

Combining all of the above we arrive at

$$|A_N| \leq |A_N\backslash B_N| \ll \frac{1}{\log N} \sum_{y\in A_N\backslash B_N} \deg y \leq \frac{1}{\log N} \sum_{x\in C_N} \deg x \ll \frac{N^2}{(\log N)^{\alpha-1+1}},$$

as wanted. $\square$

Remarks 6.2. *Using a similar argument as in the proof of Theorem 6.1, one can show that the set of large geodesic periods is small. More precisely, for any $\delta > 0$, we have*

$$\left|\left\{y \in Y_N^* : |\mathscr{P}_f(y)| \geq (\log N)^{1/2+\delta}\right\}\right| \ll_\delta \frac{N^2}{(\log N)^2}.$$

We do not need to assume the Central Limit Theorem with error term. By using the estimate for k-th moment, we have for any $A > 0$,

$$\left|\left\{x \in X_N : |L_f(x)| \geq (\log N)^{1/2+\delta}\right\}\right| \ll_{\delta,A} \frac{N^2}{(\log N)^A}.$$

We will now proceed to prove Theorems 1.4 and 1.5 from the introduction. The proof is very similar to the preceding one with the only difference being that for periods of Eisenstein series we will only have to control the contribution from $x \in X_N$ for which $c(x)$ is small.

Proof of Theorems 1.4 and 1.5. We may assume that $h(N) \leq \log\log\log N$. Put

$$\begin{aligned} A_N^- =& \left\{y \in Y_N^* : |\mathscr{P}_f(y)| \leq (\log N)^{1/2}(\log\log N)^{\delta_f}/h(N)\right\}, \\ A_N^+ =& \left\{y \in Y_N^* : |\mathscr{P}_f(y)| \geq (\log N)^{1/2}(\log\log N)^{\delta_f}h(N)\right\}, \\ A_N =& A_N^- \cup A_N^+. \end{aligned}$$

We want to show that $|A_N| = o(N^2/\log N)$ as $N \to \infty$. Let $\epsilon > 0$ and assume for contradiction that $|A_N| \geq \epsilon N^2/\log N$ for infinitely many N. For such N put

$$B_N = \{y \in Y_N : \mathrm{tr}(y) \leq N/\log\log N\}.$$

Then by the prime geodesic theorem we have for N sufficiently large

$$|A_N \setminus B_N| \geq \tfrac{1}{2}\epsilon N^2/\log N.$$

Let $C_N^{\pm} = e(A_N^{\pm} \setminus B_N)$ and $D_N = \{x \in X_N : c(x) \leq N/(\log N)^{1/2}\}$. Then by Proposition 3.1 we have that for $x \in C_N^- \setminus D_N$:

$$\begin{aligned}|L_f(x)| &\ll (\log N)^{1/2}(\log\log N)^{\delta_f}/h(N) + (\log N)^{1/4+1/100} + (\log\log N)^{1/2+1/100} \\ &\ll (\log N)^{1/2}(\log\log N)^{\delta_f}/h(N),\end{aligned}$$

and similarly for $x \in C_N^+ \setminus D_N$. Note that since $h(N) \to \infty$, we have

$$\mathbb{P}(|\mathscr{N}_{\mathbb{C}}(\sigma,\mu)| \leq h(N)^{-1}) + \mathbb{P}(|\mathscr{N}_{\mathbb{C}}(\sigma,\mu)| \geq h(N)) \to 0, \quad N \to \infty,$$

where $\mathscr{N}_{\mathbb{C}}(\sigma,\mu)$ denotes a complex normally distributed random variable with mean μ and variance σ. Thus, with $C_N = C_N^+ \cup C_N^-$, we conclude by the normal distribution results in Theorems 3.2 and 3.3 that

$$\frac{|C_N \setminus D_N|}{|X_N|} \to 0, \quad N \to \infty.$$

Let $\psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be an increasing (and sufficiently slowly growing) function such that $\psi(N) \to \infty$ and $|C_N \setminus D_N|\psi(N) = o(N^2)$ as $N \to \infty$. We proceed as above and count the number of edges between $C_N \setminus D_N$ and $A_N \setminus B_N$ in two different ways. First of all the total number of such edges is clearly upper bounded by

$$\sum_{x \in C_N \setminus D_N} \deg x = \sum_{\substack{x \in C_N \setminus D_N: \\ c_x \leq \frac{N}{\psi(N)}}} \deg x + \sum_{\substack{x \in C_N \setminus D_N: \\ c_x > \frac{N}{\psi(N)}}} \deg x \ll \frac{N^2}{\psi(N)} + |C_N \setminus D_N|\psi(N) = o(N^2).$$

On the other hand, by the assumptions on the size of $A_N \setminus B_N$ Theorem 5.8 applies and we conclude by equidistribution and the lower bound Proposition 5.4:

$$\sum_{y \in A_N \setminus B_N} \deg y \gg \sum_{y \in A_N \setminus B_N} \ell(\mathscr{C}_y \cap \mathscr{B}) \gg |A_N \setminus B_N| \log N \gg_\epsilon N^2.$$

Recall that by definition C_N is the set of neighbors of $A_N \setminus B_N$, so the number of edges between $A_N \setminus B_N$ and $C_N \setminus D_N$ is lower bounded by

$$\sum_{y \in A_N \setminus B_N} \deg y - \sum_{x \in D_N} \deg x \geq \sum_{y \in A_N \setminus B_N} \deg y - 2\frac{N^2}{(\log N)^{1/2}} \gg_\epsilon N^2.$$

This is contradiction. □

7. Non-vanishing of central values of L-functions

Denote by $\mathscr{D}_{\mathrm{fund}}$ the set of positive fundamental discriminants, that is

$$\mathscr{D}_{\mathrm{fund}} = \left\{ D > 0 : \begin{array}{l} D \equiv 1 \pmod 4 \text{ and } D \text{ square-free or} \\ D = 4m, \text{ where } m \equiv 2,3 \pmod 4 \text{ and } m \text{ square-free} \end{array} \right\}.$$

Let $D \in \mathscr{D}_{\mathrm{fund}}$ and $K = \mathbb{Q}(\sqrt{D})$ the associated real quadratic field. Let N be a square-free integer coprime with D such that each prime divisor p of N splits in K. Hence there exists an integer α such that $D \equiv \alpha^2 \pmod{4N}$. A quadratic form $Q = [a,b,c]$ is said to be *Heegner of level* N if $N|a$ and $b \equiv \alpha \pmod{2N}$. Denote by

$\mathscr{Q}_{N,D}$ the set of primitive quadratic forms Heegner of level N and discriminant D. Then $\mathscr{Q}_{N,D}$ is stable under the action of $\Gamma_0(N)$ and we have isomorphisms

$$\Gamma_0(N)\backslash\mathscr{Q}_{N,D} \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{Q}_{1,D}$$

and

$$\mathrm{Cl}_D^+ \xrightarrow{\sim} \Gamma_0(N)\backslash\mathscr{Q}_{N,D},$$

where Cl_D^+ is the narrow class group of $\mathbb{Q}(\sqrt{D})$. Therefore, for each $A \in \mathrm{Cl}_D^+$, there is an associated primitive geodesic $\mathscr{C}_A$ in $X_0(N) := \Gamma_0(N)\backslash\mathbb{H}$ of length $\log \epsilon_D^2$, we refer to [8] and [29, Chapter 6] for precise details. We denote $h(D) = |\mathrm{Cl}_D^+|$ the (narrow) class number.

For a fixed square-free N, we are interested in working with fundamental discriminants such that the above holds. A prime p splits in $\mathbb{Q}(\sqrt{D})$ iff $(p, D) = 1$ and D is quadratic residue modulo p. Therefore, from the Chinese Remainder Theorem, there exists $I \subset \{1, 2, \cdots, 4N-1\}$ such that

$$\forall p|N, p = \mathfrak{p}_1\mathfrak{p}_2 \text{ in } Q(\sqrt{D}) \Longleftrightarrow D \bmod 4N \in I.$$

We denote

$$\mathscr{D}_{\text{fund}}^N := \{D \text{ fundamental discriminant} : D \equiv a \pmod{4N}, \text{ for some } a \in I\}$$

and

$$\mathscr{D}_{\text{fund}}^N(X) := \{D \in \mathscr{D}_{\text{fund}}^N : \epsilon_D \leq X\}.$$

From work of Hashimoto [14], we know there exists a constant $a(N)$ such that

$$\sum_{D \in \mathscr{D}_{\text{fund}}^N(X)} h(D) = a(N)\mathrm{Li}(X^2) + O_{N,\epsilon}\left(X^{25/13+\epsilon}\right). \tag{7.1}$$

Let f be a Hecke–Maaß newform of weight $2k \geq 0$ for $\Gamma_0(N)$. For $D \in \mathscr{D}_{\text{fund}}^N$ and $K = Q(\sqrt{D})$, we have the following version of Waldspurger formula, given by Popa [29, Theorem 6.3.1]:

$$L(f \otimes \theta_\chi, 1/2) = \frac{c_f^+}{D^{1/2}} \left| \sum_{A \in \mathrm{Cl}_D^+} \chi(A)\mathscr{P}_f(\mathscr{C}_A) \right|^2,$$

where θ_χ is the theta-series associated to the class group character $\chi \in \widehat{\mathrm{Cl}_D^+}$ and $c_f^+ > 0$ is constant depending only on f. Note here that the geodesic periods [29, (6.1.2)] in Popa's formula do indeed match the geodesic periods $\mathscr{P}_f(\mathscr{C}_A)$ up to a constant depending only on f as follows from the explicit formulas in [29, Sec. 6.1]. By applying Plancherel (orthogonality of characters) to the above, we obtain

$$\frac{1}{h(D)} \sum_{\chi \in \widehat{\mathrm{Cl}_D^+}} L(f \otimes \theta_\chi, 1/2) = \frac{c_f^+}{D^{1/2}} \sum_{A \in \mathrm{Cl}_D^+} |\mathscr{P}_f(\mathscr{C}_A)|^2. \tag{7.2}$$

Using this together with our previous results, we obtain the following non-vanishing result for central values of Rankin-Selberg L-functions.

Proposition 7.1. *There exists an absolute constant $c = c(N) > 0$ such that, as $X \to \infty$,*

$$\frac{\#\{D \in \mathscr{D}^N_{\mathrm{fund}}(X) : \exists \chi \in \widehat{\mathrm{Cl}^+_D} \ s.t. \ L(1/2, f \otimes \theta_\chi) \neq 0\}}{\#\mathscr{D}^N_{\mathrm{fund}}(X)} \geq c + o(1). \tag{7.3}$$

Proof. Let $\mathscr{D}_{\mathrm{fund}}(X) := \{D \in \mathscr{D}_{\mathrm{fund}} : \epsilon_D \leq X\}$. We use the moments of $h(D)$ restricted to fundamental discriminants, as computed by Raulf [30]. More precisely, we have that for each $k \in \mathbb{N} \cup \{0\}$, there exists a constant $C(k)$ such that

$$\sum_{D \in \mathscr{D}_{\mathrm{fund}}(X)} h(D)^k = C(k) \int_2^X \left(\frac{t}{\log t} \right)^k dt + O\left(X^{k+1-\delta_k} \right),$$

for some $\delta_k > 0$.

We denote

$$\mathscr{D}_{\mathrm{good}}(X) := \left\{ D \in \mathscr{D}^N_{\mathrm{fund}}(X) : \exists \chi \in \widehat{\mathrm{Cl}^+_D} \text{ with } L(1/2, f \otimes \theta_\chi) \neq 0 \right\} \subset \mathscr{D}^N_{\mathrm{fund}}(X).$$

Note that if $D \in \mathscr{D}^N_{\mathrm{fund}}(X) \setminus \mathscr{D}_{\mathrm{good}}(X)$, we see from (7.2) that $\mathscr{P}_f(\mathscr{C}_A) = 0$ for all $A \in \mathrm{Cl}^+_D$. In particular,

$$\bigcup_{D \in \mathscr{D}^N_{\mathrm{fund}}(X) \setminus \mathscr{D}_{\mathrm{good}}(X)} \bigcup_{A \in \mathrm{Cl}^+_D} \{\mathscr{C}_A\} \subseteq \{\mathscr{C} \subset X_0(N) : \ell(\mathscr{C}) \leq 2 \log X, \mathscr{P}_f(\mathscr{C}) = 0\}.$$

Fix $\epsilon > 0$. Using Theorem 1.4 and the prime geodesic theorem (5.3), we obtain that for X sufficiently large

$$\sum_{D \in \mathscr{D}^N_{\mathrm{fund}}(X) \setminus \mathscr{D}_{\mathrm{good}}(X)} h(D) \leq \epsilon \frac{X^2}{\log X}.$$

Using Cauchy–Schwartz, and the fact that $\mathscr{D}_{\mathrm{good}}(X) \subseteq \mathscr{D}^N_{\mathrm{fund}}(X) \subseteq \mathscr{D}_{\mathrm{fund}}(X)$, it follows that for X sufficiently large

$$\begin{aligned} (C(2) + \epsilon) \frac{X^3}{(\log X)^2} &\geq \sum_{D \in \mathscr{D}_{\mathrm{fund}}(X)} h(D)^2 \geq \sum_{D \in \mathscr{D}_{\mathrm{good}}(X)} h(D)^2 \\ &\geq \frac{\left(\sum_{D \in \mathscr{D}_{\mathrm{good}}(X)} h(D) \right)^2}{|\mathscr{D}_{\mathrm{good}}(X)|} \geq \frac{(a(N) - \epsilon)^2 \frac{X^4}{(\log X)^2}}{\alpha X} \end{aligned}$$

This implies that for any $\epsilon > 0$ we have for X sufficiently large that

$$|\mathscr{D}_{\mathrm{good}}(X)| \geq \frac{(a(N) - \epsilon)^2}{C(2) + \epsilon}.$$

This implies that the lower bound (7.3) holds with $c = \frac{a(N)^2}{C(2)}\beta$ where

$$\beta := \liminf_{X\to\infty} \frac{X}{|\mathscr{D}^N_{\mathrm{fund}}(X)|} > 0.$$

□

Remark 7.2. We could have applied Hölder inequality instead of Cauchy–Schwartz and obtain $\alpha = \left(\frac{a(N)^k}{C(k)}\right)^{1/(k-1)}$ and obtain possibly a stronger result. However, none of the constants $C(k)$ have been computed besides $k = 0, 1$, and the upper bounds of $C(k)$ from [30] would indicate fast decay for α as $k \to \infty$.

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