The Gauss circle problem

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1 Introduction

A classical problem in mathematics, the Gauss circle problem is to find the number of integer lattice points inside the circle of radius r centered at the origin. Let $Q(r)$ be the number of lattice points inside a circle in plane of radius r, i.e.

$$Q(r) = \#\{(m, n) \in \mathbb{Z}^2 \mid m^2 + n^2 \leq r^2\} .$$

$Q(r)$ is approximated by the area of the circle, which is πr^2. Write

$$Q(r) = \pi r^2 + E(r) .$$

Hence the real problem is to accurately bound $E(r)$. The goal is to find a bound of the form

$$|E(r)| = O\left(r^\theta\right)$$

for θ as small as possible. The first result came from Gauss (1834), who showed we can have $\theta = 1$. Voronoi(1903), Sierpinski (1906) and van der Corput (1923) independently showed that we can take $\theta = 2/3$. This is nowadays called the "classical exponent". There have been many subsequent improvements, best bound known today belongs to Bourgain and Watt [BW17], who showed that that $\theta = 517/824 + \epsilon$, for all $\epsilon > 0$.

However, Hardy (1915) [Har15] and more generally Erdös, Fuchs (1956) [EF56] showed that $\theta > 1/2$, in other words

$$\limsup \frac{|E(r)|}{r^{1/2}} = \infty$$

It is conjectured that $\theta = 1/2 + \epsilon$, but all we know today is that $1/2 < \theta \leq 0.6274....$ It is interesting to note that even though there have been numerous improvements in the past 100 years, the exponent have not decreased substantially and there is a long way to go to reach the conjecture.

Sometimes, the Gauss circle problem is stated as approximating

$$G(r) = \#\{(m, n) \in \mathbb{Z}^2 \mid m^2 + n^2 \leq r\} = Q(\sqrt{r})$$

and we have that $G(r) = \pi r + E(r)$. In this setting, the classic exponent becomes $E(r) = O\left(r^{1/3}\right)$ (of course all the exponents are halved).

In this project, we will present several ways to approach the problem. First, we will study the the original argument of van der Corput which relies on the Poisson summation formula. Next we will explore how the theory of L-functions can be used to tackle our problem. In the fourth section we will expose some classical methods of van der Corput and Weyl to bound exponential sums. In the next section we will study the method of exponent pairs and we will see how we can apply it to improve the exponent in the error below 2/3. Finally, we will use van der Corput's bound for exponential sums to find an explicit constant C such that $|E(r)| \leq Cr^{2/3}$.

We would like to develop the notation used in this project. We write $A(x) = O(B(x))$ or $A(x) \ll B(x)$ when there exists an absolute constant c such that $|A(x)| \leq cB(x)$, for all values of x under consideration. When x is a real number, we let $\lfloor x \rfloor$ denote the largest integer not exceeding x, we let $\{x\}$ be the fractional part $\{x\} = x - \lfloor x \rfloor$ and we let $\psi(x) = \{x\} - 1/2$. Also, when x is a real number, we denote by $\|x\|$ the distance from x to the nearest integer. Furthermore, we use the exponential function $e(x) = e^{2\pi i x}$.

It is easy to imagine that the methods developed in this project can be used to tackle more general problems. However, for clarity and consistency reasons, and due to restrictions in the length of this report, I have chosen to present them as applied to the Gauss circle problem in the plane. The goal of this project is to present a clear and easy to understand exposure to these beautiful methods.

2 The original argument

We present the original argument of van der Corput for obtaining 2/3 in the exponent of the error. The method can be found in [Hör79].

Let $\mathbf{1}(r)$ be the characteristic function of the unit disc in plane, i.e.

$$\mathbf{1}(\mathbf{x}) = \begin{cases} 1 & \text{if } |\mathbf{x}| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

and let $\mathbf{1}_r(\mathbf{x}) = \mathbf{1}(\mathbf{x}/r)$ the characteristic function of the disc of radius r. Then

$$Q(r) = \sum_{\mathbf{x}\in\mathbb{Z}^2} \mathbf{1}_r(\mathbf{x})$$

Now let ρ be a positive smooth function on $\mathbb{R}^2$ with compact support inside the unit ball and integral one. Also, define

$$\rho_\epsilon(\mathbf{x}) = \frac{1}{\epsilon^2}\rho\left(\frac{\mathbf{x}}{\epsilon}\right)$$

so that ρ_ϵ is supported inside the ball of radius ϵ and has integral still equal to 1.

Next define

$$Q'_\epsilon(r) = \sum_{\mathbf{x}\in\mathbb{Z}^2} (\mathbf{1}_r * \rho_\epsilon)(\mathbf{x})$$

We notice that

$$\begin{aligned} (\mathbf{1}_r * \rho_\epsilon)(\mathbf{x}) = 1 &\quad \text{if } |\mathbf{x}| \leq r - \epsilon \\ (\mathbf{1}_r * \rho_\epsilon)(\mathbf{x}) = 0 &\quad \text{if } |\mathbf{x}| > r + \epsilon \\ 0 \leq (\mathbf{1}_r * \rho_\epsilon)(\mathbf{x}) \leq 1 &\quad \text{if } r - \epsilon \leq |\mathbf{x}| \leq r + \epsilon. \end{aligned}$$

Therefore

$$Q'_\epsilon(r-\epsilon) \leq Q(r) \leq Q'_\epsilon(r+\epsilon)\ . \tag{1}$$

Next we notice that

$$\begin{aligned} \widehat{\mathbf{1}_r}(\xi) &= \int_{\mathbb{R}^2} \mathbf{1}_r(\mathbf{x}) e(-\mathbf{x}.\xi)\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^2} \mathbf{1}(\mathbf{y}) e(-r\mathbf{y}.\xi) r^2\,\mathrm{d}\mathbf{y} = r^2\ \widehat{\mathbf{1}}(r\xi) \\ \widehat{\rho_\epsilon}(\xi) &= \int \rho_\epsilon(\mathbf{x}) e(-\mathbf{x}.\xi) = \int \rho(\mathbf{y}) e(-\epsilon\mathbf{y}.\xi) = \hat{\rho}(\epsilon\xi) \end{aligned}$$

Now we want to use the Poisson summation formula for $\mathbf{1}_r * \rho_\epsilon$. Indeed, we have that $\mathbf{1}_r * \rho_\epsilon$ is in the Schwartz space $S(\mathbb{R}^2)$ since the convolution of a function with compact support and a smooth function is also smooth. Also, we already showed $\mathbf{1}_r * \rho_\epsilon$ has compact support.

$$Q'_\epsilon(r) = \sum_{\mathbf{x}\in\mathbb{Z}^2} (\mathbf{1}_r * \rho_\epsilon)(\mathbf{x}) = \sum_{\mathbf{x}\in\mathbb{Z}^2} \widehat{\mathbf{1}_r * \rho_\epsilon}(\mathbf{x}) = \sum \widehat{\mathbf{1}_r}(\mathbf{x})\widehat{\rho_\epsilon}(\mathbf{x}) = r^2 \sum_{\mathbf{x}\in\mathbb{Z}^2} \widehat{\mathbf{1}}(r\mathbf{x})\widehat{\rho}(\epsilon\mathbf{x})$$

Since $\widehat{\mathbf{1}}(0) = \pi$ and $\widehat{\rho}(0) = 1$, we have that

$$Q'_\epsilon(r) = \pi r^2 + r^2 \sum_{\mathbf{x}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \widehat{\mathbf{1}}(r\mathbf{x})\widehat{\rho}(\epsilon\mathbf{x})$$

Now, since ρ is smooth with compact support, then $\rho \in S(\mathbb{R}^2)$. This implies that also $\widehat{\rho} \in S(\mathbb{R}^2)$, which means that $|\widehat{\rho}(\mathbf{x})| \ll_N (1 + |\mathbf{x}|^2)^{-N}$, for all positive integers N.

We claim that $\left|\widehat{\mathbf{1}}(\mathbf{x})\right| \ll |\mathbf{x}|^{-3/2}$. Indeed,

$$\widehat{\mathbf{1}}(\mathbf{x}) = \int_{|\mathbf{y}|\leq 1} e^{-2\pi i(\mathbf{x}.\mathbf{y})}\,\mathrm{d}\mathbf{y} = \int_0^1 \int_0^{2\pi} r e(-|\mathbf{x}| r\cos(\theta - \alpha))\,\mathrm{d}\theta\,\mathrm{d}r = \frac{J_1(2\pi|\mathbf{x}|)}{|\mathbf{x}|}$$

where the first equality comes from change to polar coordinates and J_1 is the Bessel function of the first kind

$$J_1(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(y-\sin y)}\,\mathrm{d}y$$

We have the asymptotic formula from [AS64, p. 364]

$$J_1(|\mathbf{x}|) = \sqrt{\frac{2}{\pi|\mathbf{x}|}}\left(\cos\left(|\mathbf{x}| - \frac{3\pi}{4}\right) + O(|\mathbf{x}|^{-1})\right)$$

which completes the proof of the claim (the claim can also be proved using the stationary phase lemma). Now we approximate the error term:

$$r^2 \sum_{\mathbf{x}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \widehat{\mathbf{1}}(r\mathbf{x})\widehat{\rho}(\epsilon\mathbf{x}) \ll r^{1/2} \sum_{\mathbf{x}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} |\mathbf{x}|^{-3/2}(1+\epsilon^2|\mathbf{x}|^2)^{-N} \ll r^{1/2}\int_{\mathbb{R}^2} \frac{(1+\epsilon^2|\mathbf{x}|^2)^{-N}}{|\mathbf{x}|^{3/2}}\,\mathrm{d}\mathbf{x}$$
$$\ll r^{1/2}\epsilon^{-1/2}\int_{\mathbb{R}^2} \frac{(1+|\mathbf{y}|^2)^{-N}}{|\mathbf{y}|^{3/2}}\,\mathrm{d}\mathbf{y} \ll r^{1/2}\epsilon^{-1/2}$$

if we assume $N \geq 2$.

Next we take $\epsilon = r^{-1/3}$, so we have that $Q'_\epsilon(r) = \pi r^2 + O(r^{2/3})$. This implies that

$$Q'_\epsilon(r+\epsilon) = \pi(r + r^{-1/3})^2 + O((r + r^{-1/3})^{2/3}) = \pi r^2 + O(r^{2/3}) \tag{2}$$

We obtain a similar estimate for $Q'_\epsilon(r-\epsilon)$. Combining this with (1), we obtain that $Q(r) = \pi r^2 + O(r^{2/3})$.
$\blacksquare$

3 L-functions approach

Let $r(n)$ be the number ways of writing n as a sum of two integer squares. Then clearly we have

$$G(x) = \sum_{n\leq x} r(n)$$

Let χ the non-trivial character modulo 4, i.e.

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \mod 4 \\ -1 & \text{if } n \equiv 3 \mod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.1

$$r(n) = 4\sum_{d|n}\chi(d)\ .$$

Proof Let

$$\delta(n) = \sum_{d|n}\chi(d) = d_1(n) - d_3(n)$$

where $d_1(n)$ and $d_3(n)$ are the numbers of divisors of n of the forms $4m+1$ and $4m+3$ respectively. Say

$$n = 2^a \prod_{j=1}^{k} p_j^{b_j} \prod_{j=1}^{l} q_j^{c_j}$$

where p_j are primes of the form $4m+1$ and q_j are primes of the form $4m+3$. Then it is easy to see that

$$\delta(n) = \prod_{j=1}^{k}(b_j+1)\prod_{j=1}^{l}\left(\frac{1+(-1)^{c_j}}{2}\right)$$

In particular, $\delta(n) = 0$ if not all of c_j are even and $\prod(b_j+1)$ otherwise. Now say $n = A^2 + B^2 = (A+Bi)(A-Bi)$ and let's look at their factorisation in $\mathbb{Z}[i]$:

$$A + Bi = i^r(1+i)^{a_1}(1-i)^{a_2}\prod_{j=1}^{k}(x_j+iy_j)^{\beta_{j1}}(x_j-iy_j)^{\beta_{j2}}\prod_{j=1}^{l}q^{\gamma_j}$$
$$A - Bi = i^{-r}(1+i)^{a_2}(1-i)^{a_1}\prod_{j=1}^{k}(x_j+iy_j)^{\beta_{j2}}(x_j-iy_j)^{\beta_{j1}}\prod_{j=1}^{l}q^{\gamma'_j}$$

where $a_1 + a_2 = a$, $\beta_{j1} + \beta_{j2} = b_j$, $\gamma_j + \gamma'_j = c_j$ and $p_j = (x_j + iy_j)(x_j - iy_j)$ is the unique factorisation of a prime of the form $4m+1$ in $\mathbb{Z}[i]$. If we look at the norms, we must have that $\gamma_j = \gamma'_j = c_j/2$. Next, since $(1-i)/(1+i) = -i$ is a unit, the choice of a_1 or a_2 produces no variation in A and B beyond that produced by the choice of r. Hence the total number of representations is $4\prod(b_j+1) = 4\delta(n)$. ∎

We define

$$Z(s) = \sum_{n\geq 1}\frac{r(n)}{n^{-s}} \tag{3}$$

which is a L-function well defined on $\mathrm{Re}(s) > 1$. We see that

$$Z(s) = \sum_{n\geq 1}\frac{r(n)}{n^{-s}} = 4\sum_{n\geq 1}n^{-s}\sum_{d|n}\chi(d) = 4\sum_{d\geq 1}\chi(d)d^{-s}\sum_{m\geq 1}m^{-s} = 4\zeta(s)L(\chi,s)$$

where we used absolute convergence for $\mathrm{Re}(s) > 1$. This gives us analytic continuation to all of $\mathbb{C}$ with one simple pole at $s = 1$ with residue π (since $L(\chi, 1) = \pi/4$).

Recall the functional equations for $\zeta(s)$ and $L(\chi, s)$ from [Dav00, Chapter 9]

$$\zeta(s) = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$$
$$L(\chi,s) = 2^{1-s}\pi^{s-1}\sin\left(\frac{\pi}{2}(s+1)\right)\Gamma(1-s)\mathrm{L}(\chi,1-s)$$

Hence we obtain

$$Z(s) = \pi^{2(s-1)}\sin(\pi s)(\Gamma(1-s))^2 Z(1-s) \tag{4}$$

For notational convenience, let

$$\alpha(s) = \pi^{2(s-1)}\sin(\pi s)(\Gamma(1-s))^2$$

and we have $Z(s) = \alpha(s)Z(1-s)$.

We want to find a formula for $G(x) = \sum_{n\leq x} r(n)$ using $Z(s)$. For this purpose, we begin by recalling the Perron formula. For $c > 0$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}\,\mathrm{d}s = \begin{cases}1 & \text{if } x > 1\\ 1/2 & \text{if } x = 1\\ 0 & \text{if } 0 < x < 1.\end{cases}$$

Hence, for $c > 0$, $\mathrm{Re}(s) > 1$ and $x > 0$ not an integer:

$$G(x) = \sum_{n<x}r(n) = \sum_{n=1}^{\infty}r(n)\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(x/n)^s}{s}\,\mathrm{d}s = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}Z(s)\frac{x^s}{s}\,\mathrm{d}s \tag{5}$$

We want to use something stronger than the Perron formula in order to obtain better approximations for $G(x)$. We adapt the approach from [Tit86, Chapter 12]. We will show that

$$E(r) =\ll_\epsilon r^{1/3+\epsilon}\ .$$

Before we start, we need the following lemma:

Lemma 3.2 *[Tit86, Lemma 4.5] Let $F(x)$ be a twice differentiable real function such that $|F"(x)| \geq r > 0$ on the interval $[a,b]$. Let $H(x)$ be real a real function such that $H(x)/F'(x)$ monotonic and $|H(x)| \leq M$ on $[a,b]$. Then*

$$\left|\int_a^b H(x)e^{iF(x)}\,\mathrm{d}x\right| \leq \frac{8M}{\sqrt{r}}\ .$$

Proof We assume without losing the generality that $F"(x) \geq r > 0$. Then F' is a strictly increasing function, so it vanishes at most once, say at c. Let δ a parameter to be chosen later.

We first consider the case $a + \delta \leq c \leq b - \delta$. Then

$$I = \int_a^b H(x) e^{iF(x)} \, \mathrm{d}x = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b = I_1 + I_2 + I_3.$$

For $x \geq c + \delta$, $F'(x) = \int_c^x F"(t) \, \mathrm{d}t \geq r(x - c) \geq r\delta$. Then

$$\begin{aligned} \operatorname{Re} I_3 &= \int_{c+\delta}^b H(x) \cos F(x) \, \mathrm{d}x = \int_{c+\delta}^b \frac{H(x)}{F'(x)} F'(x) \cos F(x) \, \mathrm{d}x \\ &= \frac{H(y)}{F'(y)} \int_{c+\delta}^b F'(x) \cos F(x) \, \mathrm{d}x = \frac{H(y)}{F'(y)} (\sin F(b) - \sin(F(c+\delta)) \end{aligned}$$

for some $y \in [c + \delta, b]$, since $H(x)/F'(x)$ monotonic. Hence $|\operatorname{Re} I_3| \leq 2M/(r\delta)$. We obtain the same bound for the imaginary part, so $|I_3| \leq 4M/(r\delta)$.

Similarly, $|I_1| \leq 4M/(r\delta)$. Also, it is easy to see that $|I_2| \leq 2\delta M$. Hence

$$|I| \leq \frac{8M}{r\delta} + 2M\delta$$

Choose $\delta = 2r^{-1/2}$ and the conclusion follows.

For the other cases, proceed similarly, split the integral into one or two or three depending on $(c - \delta, c + \delta) \cap [a, b]$. ∎

We are now ready to start. We use the following lemma:

Lemma 3.3 *[Tit86, Lemma 3.12] Let* $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ *where* $a_n = O(\psi(n))$ *and*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} = O\left(\frac{1}{(\sigma - 1)^\alpha} \right)$$

as $\sigma \to 1$. *Then, if* $c > 0$, $\sigma + c > 1$, x *not an integer,* N *the nearest integer to* x, *then*

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} \, \mathrm{d}w + O\left(\frac{x^c}{T(\sigma + c - 1)^2} \right) + O\left(\frac{\psi(2x) x^{1-\sigma} \log x}{T} \right) + O\left(\frac{\psi(N) x^{1-\sigma}}{T\|x\|} \right).$$

We omit the proof due to constraints in the length of the project. The proof is just an application of the residue theorem for a suitably chosen function and domain of integration.

Applying it with $a_n = r(n)$, $\psi(n) = n^\epsilon$, $\alpha = 2$, $s = 0$, $c = 1+\epsilon$ and x half an odd integer, we obtain

$$G(x) = \frac{1}{2\pi i} \int_{1+\epsilon-iT}^{1+\epsilon+iT} Z(s) \frac{x^s}{s} \, \mathrm{d}s + O\left(\frac{x^{1+\epsilon}}{T\epsilon^2} \right) + O\left(\frac{x^{1+\epsilon}}{T} \right) \tag{6}$$

We adapt the approach from [Tit86, Theorem 12.2]. Let

$$I = \frac{1}{2\pi i} \left(\int_{1+\epsilon-iT}^{1+\epsilon+iT} Z(s) \frac{x^s}{s} \, \mathrm{d}s + \int_{1+\epsilon+iT}^{-\epsilon+iT} Z(s) \frac{x^s}{s} \, \mathrm{d}s + \int_{-\epsilon+iT}^{-\epsilon-iT} Z(s) \frac{x^s}{s} \, \mathrm{d}s + \int_{-\epsilon-iT}^{1+\epsilon-iT} Z(s) \frac{x^s}{s} \, \mathrm{d}s \right)$$

By the residue theorem, we notice that

$$I = \operatorname{Res}_{s=0} Z(s) \frac{x^s}{s} + \operatorname{Res}_{s=1} Z(s) \frac{x^s}{s} = Z(0) + \pi x = \pi x + O(1)$$

Using Sterling's formula for the Γ function [Dav00, Chapter 10], we have that

$$\log \Gamma(s) = (s - 1/2) \log s - s + \frac{1}{2} \log 2\pi + (|s|^{-1})$$

valid in the angle $-\pi - \delta < \arg s < \pi - \delta$ as $|s| \to \infty$, for any fixed $\delta > 0$. This implies that, in a fixed strip $\alpha \leq \sigma \leq \beta$, as $t \to \infty$, (c.f. 4.12.2 in [Tit86])

$$\Gamma(\sigma + it) = (2\pi)^{1/2} t^{\sigma+it-1/2} e^{-\frac{1}{2}\pi t - it + \frac{1}{2} i\pi(\sigma-1/2)} \left(1 + O\left(\frac{1}{t}\right)\right) \tag{7}$$

Using the functional equation for Γ function [Dav00, Chapter 10]

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

and that, as $t \to \infty$,

$$\sin(\sigma + it) \sim \frac{ie^t}{2} e^{-i\sigma} ,$$

we obtain that

$$\alpha(\sigma + it) = -i \left(\frac{\pi}{t}\right)^{2(\sigma+it)-1} e^{i(2t+\frac{1}{2}\pi)} \left(1 + O\left(\frac{1}{t}\right)\right) \tag{8}$$

as $t \to \infty$. Since $Z(s)$ is bounded in the half plane $\sigma \geq 1 + \epsilon$, using the functional equation for Z we obtain that as $t \to \infty$

$$|Z(-\epsilon + it)| \sim \left(\frac{t}{\pi}\right)^{1+2\epsilon}$$

We define $\mu(\sigma)$ the infimum of all real numbers a such that $Z(\sigma + it) = O(t^a)$, for all t. It is a classical application of the Phragmen-Lindelöf principle to show that μ is a convex function (see [Lan99] for more details). Since $\mu(-\epsilon) \leq 1 + 2\epsilon$ and $\mu(1+\epsilon) \leq 0$, we have that, for $-\epsilon \leq \sigma \leq 1 + \epsilon$

$$Z(\sigma + it) = O\left(t^{(1+2\epsilon)(1+\epsilon-\sigma)/(1+2\epsilon)}\right)$$

Hence

$$\int_{-\epsilon+iT}^{1+\epsilon+iT} Z(s)\frac{x^s}{s}\,\mathrm{d}s \ll \int_{-\epsilon}^{1+\epsilon} T^{(1+2\epsilon)(1+\epsilon-\sigma)/(1+2\epsilon)-1} x^\sigma \ll T^{2\epsilon}x^{-\epsilon} + T^{-1}x^{1+\epsilon}\,\mathrm{d}\sigma$$

since the integrand is maximum at one end or the other of the integral. We obtain a similar estimate for the integral from $-\epsilon - iT$ to $1 + \epsilon - iT$.

Next,

$$\int_{-\epsilon-iT}^{-\epsilon+iT} Z(s)\frac{x^s}{s}\,\mathrm{d}s = \int_{-\epsilon-iT}^{-\epsilon+iT} \alpha(s)Z(1-s)\frac{x^s}{s}\,\mathrm{d}s = \sum_{n=1}^{\infty} r(n) \int_{-\epsilon-iT}^{-\epsilon+iT} \frac{\alpha(s)}{n^{1-s}}\frac{x^s}{s}\,\mathrm{d}s$$
$$= ix^{-\epsilon} \sum_{n=1}^{\infty} \frac{r(n)}{n^{1+\epsilon}} \int_{-T}^{T} \frac{\alpha(-\epsilon+it)}{-\epsilon+it}(nx)^{it}\,\mathrm{d}t$$

We are interested to approximate the interior integral for $1 \leq t \leq T$. We note that

$$\frac{1}{-\epsilon + it} = \frac{1}{it} + O\left(\frac{1}{t^2}\right) .$$

Combining this with (8), we obtain that

$$\int_1^T \frac{\alpha(-\epsilon+it)}{-\epsilon+it}(nx)^{it}\,\mathrm{d}t = -i\pi^{-2\epsilon-1} \int_1^T e^{2it(\log\pi - \log t + 1}(nx)^{it} t^{2\epsilon}\,\mathrm{d}t + O(T^{2\epsilon})$$

Let $F(t) = -2t(\log \pi - \log t + 1) + t \log nx$ and $H(t) = t^{2\epsilon}$. Now $F"(t) = 2/t \geq 2/T$ and

$$\left(\frac{H(t)}{F'(t)}\right)' = \frac{2t^{2\epsilon-1}(-2\epsilon\log\pi + 2\epsilon\log t + \epsilon\log nx - 1)}{F'(t)^2} > 0$$

for x large enough. Hence we satisfy the conditions in lemma 3.2, so we obtain

$$\int_1^T e^{2it(\log\pi - \log t + 1)}(nx)^{it}t^{2\epsilon}\,\mathrm{d}t = \int_1^T e^{iF(t)}H(t)\,\mathrm{d}t \ll T^{1/2+2\epsilon}$$

We obtain a similar bound for the integral from $-T$ to 1. Clearly, the integral from -1 to 1 is bounded. Hence,

$$\int_{-\epsilon-iT}^{-\epsilon+iT} Z(s)\frac{x^s}{s}\,\mathrm{d}s \ll x^{-\epsilon}\sum_{n=1}^{\infty}\frac{r(n)}{n^{1+\epsilon}}T^{1/2+2\epsilon} \ll Z(1+\epsilon)x^{-\epsilon}\ T^{1/2+2\epsilon}$$

Finally, putting everything together, we obtain

$$G(x) = \pi x + O\left(\frac{x^{1+\epsilon}}{T\epsilon^2}\right) + O(T^{2\epsilon}x^{-\epsilon}) + O(T^{-1}x^{1+\epsilon}) + O(Z(1+\epsilon)x^{-\epsilon}T^{1/2+2\epsilon})$$

Choosing $T = x^{2/3}$, we get

$$E(x) \ll_\epsilon x^{1/3+\epsilon}$$

as desired. Clearly the restriction of x being half an odd integer is unnecessary to the result. ∎

4 Exponential sums

In this section, we will develop some methods of Weyl and van der Corput for bounding exponential sums. As we will see shortly see, exponential sums play an essential role in the Gauss circle problem. Actually, exponential sums are an important recurring theme in analytic number theory, important examples being finding estimates $\zeta(z)$ or the Vinogradov circle method.

The section is mainly inspired from [IK04, Chapter 8], [GK91, Chapter 2] and [Tit86, Chapter V]. We will give explicit bounds for the first few basic estimates (as we will use them later in finding an absolute constant), but we will be rather relaxed for the rest of the section.

From now on, let $I = [A, B]$ be an interval, where A and B are integers. Define $|I| = B-A+1$ and

$$S = \sum_{n=A}^{B} e(f(n))$$

4.1 Basic estimates

Theorem 4.1 *(Kusmin-Landau)*
Let $f : [A, B] \to \mathbb{R}$ be a continuously differentiable function such that f' monotone and $\|f'\| \geq \lambda > 0$ on $[A, B]$. Then

$$|S| \leq \frac{1}{\lambda}\ .$$

Proof Since $|\sum_{n=A}^{B} e(f(n))| = |\sum_{n=A}^{B} e(-f(n))|$, we may assume without losing the generality that f' is increasing (otherwise replace f by $-f$).

Let $g(n) = f(n+1) - f(n) = f'(x_n)$, for some $x_n \in [n, n+1]$, by the mean value theorem. Hence $g(n)$ is increasing on the integers in $[A, B]$. We may assume without losing the generality that $g(n) \in [\lambda, 1-\lambda]$, for all n.

Let

$$d_n = \frac{1}{1-e(g(n))}\ .$$

Then $e(f(n)) - e(f(n+1)) = e(f(n))(1 - e(g(n)))$, therefore $e(f(n)) = (e(f(n)) - e(f(n+1)))d_n$. Also,

$$d_n = \frac{1}{1-e(g(n))} = \frac{e(-g(n)/2)}{e(-g(n)/2) - e(g(n)/2)} = \frac{\cos(\pi g(n)) - i\sin(\pi g(n))}{-2i\sin(\pi g(n))} = \frac{1}{2}(1 + i\cot(\pi g(n))).$$

This implies that

$$|d_n| = |1-d_n| = \frac{1}{2\sin(\pi g(n))}\ ,\ \text{for all } n\ .$$

Hence

$$\sum_{n=A}^{B} e(f(n)) = \sum_{n=A}^{B-1} (e(f(n)) - e(f(n+1)))d_n + e(f(B))$$

$$= e(f(A))d_A + \sum_{n=A+1}^{B-1} e(f(n))(d_n - d_{n-1}) + e(f(B))(1 - d_{B-1})$$

Therefore

$$\left|\sum_{n=A}^{B} e(f(n))\right| \leq \frac{1}{2\sin \pi g(A)} + \sum_{n=A+1}^{B-1} \frac{1}{2}|\cot \pi g(n) - \cot \pi g(n-1)| + \frac{1}{2\sin \pi g(B)}$$

Since $\cot(\pi g(n))$ is a decreasing, the absolute values bars in the sum can be removed, so

$$\left|\sum_{n=A}^{B} e(f(n))\right| \leq \frac{1}{2\sin \pi g(A)} + \frac{\cos \pi g(A)}{2\sin \pi g(A)} - \frac{\cos \pi g(B-1)}{2\sin \pi g(B-1)} + \frac{1}{2\sin \pi g(B)}$$

$$\leq \frac{1}{\sin \pi g(A)} + \frac{1}{\sin \pi g(B)} \leq \frac{2}{\sin \pi\lambda} \leq \frac{1}{\lambda}$$

since $\sin(\pi\lambda) \geq 2\lambda$ for $\lambda \in (0, 1/2]$. ∎

Remark Theorem 4.1 is sharp (consider $f(x) = x$ for example), but the hypothesis is rather restrictive. We relax it in the following theorem:

Theorem 4.2 *(van der Corput)*
Let $f : [A, B] \to \mathbb{R}$ twice continously differentiable such that $0 < \lambda \leq |f"(x)| \leq h\lambda$ on $[A, B]$, for some $h > 1$. Then

$$|S| \leq 4h(B - A + 1)\lambda^{1/2} + 8\lambda^{-1/2} .$$

Proof We fix a parameter $0 < \delta < 1/2$ to be chosen later. We want to split $[A, B]$ into intervals of type (I) on which $\|f"\| \geq \delta$ and intervals of type (II) on which $\|f'\| < \delta$. Suppose without losing the generality that $f" > 0$ on $[A, B]$ and so f' increasing. Let $f'(A) = \alpha$ and $f'(B) = \beta$. Then there are $\leq \beta - \alpha + 2$ intervals of type (I) and $\leq \beta - \alpha + 2$ intervals of type (II). All intervals of type (II) are of length $\leq 2\delta/\lambda$ (since on such an interval $f'(x) \in (n - \delta, n + \delta)$, for some n, and $f" > \lambda$).

Now we use the previous theorem on all intervals of type (I) to obtain

$$\left|\sum_{n=A}^{B} e(f(n))\right| \leq (\beta - \alpha + 2)(\delta^{-1} + 2\delta\lambda^{-1} + 1) \tag{9}$$

But we must have $\beta - \alpha \leq h\lambda(B - A)$. Note that the conclusion is trivial if $\lambda \geq 1/2$, so we may assume that $\lambda < 1/2$. Take $\delta = \sqrt{\lambda/2}$ and then the conclusion follows. ∎

4.2 The Weyl - van der Corput inequality

Define $I(h) = \{n \in \mathbb{Z} \ : \ n \in I \text{ and } n + h \in I\}$.

Define

$$S_1(h) = \sum_{n \in I(h)} e(f(n + h) - f(n)) .$$

Lemma 4.3 *(Weyl - van der Corput) Let be a positive integer. Then*

$$|S|^2 \leq \frac{|I| + H}{H} \sum_{|h|<H} |S_1(h)| . \tag{10}$$

Proof In order to make the notation easier, let's define

$$\xi(n) = \begin{cases} e(f(n) & \text{if } n \in I \\ 0 & \text{otherwise} \end{cases}$$

Then it is easy to see that

$$H\sum_{n\in\mathbb{Z}}\xi(n)=\sum_{k=1}^{H}\sum_{n}\xi(n+k)=\sum_{n}\sum_{k=1}^{H}\xi(n+k)$$

and the inner sum is empty unless $A-H\le n\le B-1$. Applying Cauchy-Schwarz, we get

$$\begin{aligned}H^2|S|^2&=\left|\sum_{n=A-H}^{B-1}\left(\sum_{k=1}^{H}\xi(n+k)\right)\right|^2\le(|I|+H)\sum_{n}\left|\sum_{k=1}^{H}\xi(n+k)\right|^2\\&=(|I|+H)\sum_{n}\sum_{k=1}^{H}\sum_{l=1}^{H}\xi(n+k)\overline{\xi(n+l)}\\&=(|I|+H)\sum_{|h|<H}(H-|h|)\sum_{n}\xi(n+h)\overline{\xi(n)}\\&=(|I|+H)\sum_{|h|<H}(H-|h|)S_1(h)\end{aligned}$$

The conclusion follows easily. ∎

It is easy to see that $S_1(-h)=\overline{S_1(h)}$ and also that $S_1(h)=0$ for $h\ge|I|$. Hence, under the natural assumption that $H\le|I|$, using that $S_1(0)=|I|$, we have that

$$|S|^2\le\frac{2|I|^2}{H}+\frac{4|I|}{H}\sum_{1\le h\le H}|S_1(h)| \tag{11}$$

Also, if I' is any interval containing I and $H\le|I'|$, then

$$|S|^2\le\frac{2|I'|^2}{H}+\frac{4|I'|}{H}\sum_{1\le h\le H}|S_1(h)| \tag{12}$$

This form of the inequality is weaker and looks redundant, but it will be useful later when we will iterate S_1.

Next we provide an improvement to theorem 4.2 assuming f is thrice differentiable:

Theorem 4.4 *Let $f:I\to\mathbb{R}$ with three continuous derivatives and suppose there exists $\lambda>0$ and $\alpha\ge1$ such that $\lambda\le\left|f^{(3)}(x)\right|\le\alpha\lambda$. Then*

$$|S|\ll|I|\lambda^{1/6}\alpha^{1/3}+|I|^{3/4}\alpha^{1/4}+|I|^{1/4}\lambda^{-1/4}$$

Proof Define $f_h:[A,B-h]\to\mathbb{R}$ given by $f_h(n)=f(n+h)-f(n)$. Then clearly

$$f_h"(n)=\int_n^{n+h}f^{(3)}(x)\,\mathrm{d}x=\int_0^1 hf^{(3)}(n+xh)\,\mathrm{d}x$$

Hence $h\lambda\le|f_h"(n)|\le\alpha h\lambda$. Hence we can apply theorem 4.2 to obtain:

$$|S_1(h)|\ll|I|\alpha h^{1/2}\lambda^{1/2}+h^{-1/2}\lambda^{-1/2}$$

Combining this with (11) to obtain

$$|S|^2\ll|I|^2H^{-1}+|I|^2\alpha H^{1/2}\lambda^{1/2}+|I|H^{-1/2}\lambda^{-1/2}$$

This is true for all $0\le H\le|I|$. Of course, we want to remove the dependency on H, i.e. to minimise the left hand side. For this purpose, we apply the handy lemma 4.5 with $a_1=1/2$, $b_1=1$, $b_2=1/2$, $A_1=|I|^2\lambda^{1/2}$, $B_1=|I|^2$ and $B_2=|I|\lambda^{-1/2}\alpha$ to obtain

$$|S|^2\ll|I|^2\lambda^{1/3}\alpha^{2/3}+|I|^{3/2}\alpha^{1/2}+|I|+|I|^{1/2}\lambda^{-1/2}$$

Since the second term dominates the term and $3\sqrt{a^2+b^2+c^2}\ge a+b+c$, where a,b,c are positive reals, the conclusion follows. ∎

We used the following lemma in the previous proof:

Lemma 4.5 *Let $F(x) = \sum_{i=1}^m A_i x^{a_i} + \sum_{j=1}^n B_j x^{-b_j}$, where A_i, B_j, a_i, b_j are positive. Let $0 \le M < N$. Then there exists $x_0 \in [M, N]$ such that*

$$F(x_0) \le (m+n) \left(\sum_{i=1}^m \sum_{j=1}^n (A_i^{b_j} B_j^{a_i})^{1/a_i+b_j} + \sum_{i=1}^m A_i M^{a_i} + \sum_{j=1}^n B_j N^{-b_j} \right) .$$

Proof Define $F_+(x) = \max(A_1 x^{a_1}, \ldots, A_m x^{a_m})$ and $F_-(x) = \max(B_1 x^{-b_1}, \ldots, B_n x^{-b_n})$. Clearly $F(x) \le mF_+(x) + nF_-(x)$. We see that F_+ is increasing continuous, $F_+(0) = 0$ and $\lim_{x\to\infty} F_+(x) = \infty$. Also F_- is decreasing continuous, $\lim_{x\to 0} F_-(x) = \infty$ and $\lim_{x\to\infty} F_-(x) = 0$. Hence there exists unique $x_0 > 0$ such that $F_+(x_0) = F_-(x_0)$. We have three cases:

- $M \le x_0 \le N$
  Then there exists some i and j such that $A_i x_0^{a_i} = B_j x_0^{-b_j}$, therefore $x_0 = (B_j/A_i)^{1/(a_i+b_j)}$ and
  $$F_+(x_o) = F_-(x_0) = \left(A_i^{b_j} B_j^{a_i}\right)^{1/(a_i+b_j)}$$
- $x_0 < M$, then $F_-(M) \le F_+(M)$ and we take $x_0 = M$
- $x_0 > N$, then $F_+(N) \le F_-(N)$ and we take $x_0 = N$. ∎

4.3 Iterating van der Corput

We would like to iterate the definition of $S_1(h)$. For this purpose, we define

$$\begin{aligned} I(h_1, h_2) &= \{n \in \mathbb{Z} \;:\; n, n+h_1, n+h_2, n+h_1+h_2 \in I\} \\ f_2(n; h_1, h_2) &= f(n+h_1+h_2) - f(n+h_1) - f(n+h_2) + f(n), \text{ for } n \in I(h_1, h_2) \\ S_2(h_1, h_2) &= \sum_{n \in I(h_1,h_2)} e(f_2(n; h_1, h_2)) \end{aligned}$$

Next, applying (12), we see that for $H \le |I|$

$$|S_1(h_1)|^2 \le \frac{2|I|^2}{H} + \frac{4|I|}{H} \sum_{1\le h_2 \le H} |S_2(h_1, h_2)| \tag{13}$$

Now, using (11) and Cauchy-Schwarz, we have that

$$\begin{aligned} |S|^4 &\le \left(\frac{2|I|^2}{H_1} + \frac{4|I|}{H_1} \sum_{1\le h_1\le H_1} |S_1(h_1)| \right)^2 \\ &\le \frac{8|I|^4}{H_1^2} + \frac{32|I|^2}{H_1^2} \left(\sum_{1\le h_1\le H_1} |S_1(h_1)| \right)^2 \\ &\le \frac{8|I|^4}{H_1^2} + \frac{32|I|^2}{H_1} \sum_{1\le h_1\le H_1} \left(\frac{2|I|^2}{H_2} + \frac{4|I|}{H_2} \sum_{1\le h_2\le H} |S_2(h_1, h_2)| \right) \\ &\le \frac{8|I|^4}{H_1^2} + \frac{64|I|^4}{H_2} + \frac{128|I|^3}{H_1 H_2} \sum_{1\le h_1\le H_1} \sum_{1\le h_2\le H_2} |S_2(h_1, h_2)| \end{aligned}$$

for $H_1, H_2 \le |I|$. To make notation easier, suppose we have that $H_1^2 = H_2$ to obtain

$$|S|^4 \le \frac{128|I|^4}{H_2} + \frac{128|I|^3}{H_1 H_2} \sum_{1\le h_1\le H_1} \sum_{1\le h_2\le H_2} |S_2(h_1, h_2)| \tag{14}$$

We want to generalise for higher iterations. Define

$$\begin{aligned} I(h_1,\dots,h_k) &= \{n\in\mathbb{Z}\ : n\in I \text{ and } n+h_1+h_2+\dots+h_k\in I\} \\ f_k(n;h_1,\dots h_k) &= f_{k-1}(n+h_k;h_1,\dots,h_{k-1}) - f_{k-1}(n;h_1,\dots,h_{k-1}) \\ &= \int_0^1\cdots\int_0^1 \frac{\partial^k}{\partial x_1\dots\partial x_k} f(n+h_1x_1+\dots+h_kx_k)\,\mathrm{d}x_1\dots\mathrm{d}x_k \\ S_k(h_1,\dots,h_k) &= \sum_{n\in I(h_1,\dots,h_k)} e(f_k(n;h_1,\dots,h_k)) \end{aligned}$$

We provide an analogue for (14) in general case:

Lemma 4.6 *Let n be a positive integer and let $H_1,\dots H_n$ be such that*

$$H_1^{2^{n-1}} = H_2^{2^{n-2}} = \dots = H_n = H \le |I|\ .$$

Let $N=2^n$. Then

$$|S|^N \le 8^{N-1}\left(\frac{|I|^N}{H_n} + \frac{|I|^{N-1}}{H_1H_2\dots H_n}\sum_{1\le h_1\le H_1}\cdots\sum_{1\le h_n\le H_n}|S_n(h_1,\dots,h_n)|\right) \tag{15}$$

Proof Of course, the proof will be by induction. We already proved the base case for $n=1,2$. Now suppose it is true for n. Then

$$\begin{aligned} |S|^{2^{n+1}} &\le 8^{2N-2}\left(\frac{|I|^N}{H_n^2} + \frac{|I|^{N-1}}{H_1H_2\dots H_n}\sum_{1\le h_1\le H_1}\cdots\sum_{1\le h_n\le H_n}|S_n(h_1,\dots,h_n)|\right)^2 \\ &\le 8^{2N-2}\left(\frac{2|I|^{2N}}{H_n^2} + \frac{2|I|^{2N-2}}{H_1H_2\dots H_n}\sum_{1\le h_1\le H_1}\cdots\sum_{1\le h_n\le H_n}|S_n(h_1,\dots,h_n)|^2\right) \\ &\le 8^{2N-2}\left(\frac{2|I|^{2N}}{H_n^2} + \frac{2|I|^{2N-2}}{H_1H_2\dots H_n}\sum_{h_1,\dots h_n}\left(\frac{2|I|^2}{H_{n+1}} + \frac{4|I|}{H_{n+1}}\sum_{1\le h_{n+1}\le H_{n+1}}|S_{n+1}(h_1,\dots h_{n+1})|\right)\right) \\ &\le 8^{2N-2}\left(\frac{2|I|^{2N}}{H_n^2} + \frac{4|I|^{2N}}{H_{n+1}} + \frac{8|I|^{2N-1}}{H_1H_2\dots H_{n+1}}\sum_{h_1,\dots h_{n+1}}|S_{n+1}(h_1,\dots h_{n+1})|\right) \\ &\le 8^{2N-1}\left(\frac{|I|^{2N}}{H_{n+1}} + \frac{|I|^{2N-1}}{H_1H_2\dots H_{n+1}}\sum_{h_1,\dots h_{n+1}}|S_{n+1}(h_1,\dots h_{n+1})|\right) \end{aligned}$$

which completes the proof by induction. ∎

Next we provide a generalisation of theorem 4.4.

Theorem 4.7 *Let k be a positive integer and $f: I\to\mathbb{R}$ with $k+2$ continuous derivatives such that there exists $\lambda>0$ and $\alpha\ge 1$ such that*

$$\lambda \le \left|f^{(k+2)}(x)\right| \le \alpha\lambda\ .$$

Let $K=2^k$. Then

$$|S| \ll (\alpha\lambda)^{1/(4K-2)}|I| + \alpha^{1/2K}|I|^{1-1/2K} + \lambda^{-1/2K}|I|^{1-2/K+1/K^2}$$

Proof From the definition of $f_k(n;h_1,\dots,h_k)$ we see that

$$f_k(n;h_1,\dots,h_k) = h_1h_2\dots h_k\int_0^1\cdots\int_0^1 f^{(k)}(n+h_1x_1+h_2x_2+\dots+h_kx_k)\,\mathrm{d}x_1\dots\mathrm{d}x_k$$

therefore

$$\lambda h_1h_2\dots h_k \le |f_k(n;h_1,\dots,h_k)| \le \alpha\lambda h_1\dots h_k$$

so from theorem 4.2 it follows that

$$|S_k(h_1,\ldots,h_k)| \ll \alpha|I|(\lambda h_1 h_2 \ldots h_k)^{1/2} + (\lambda h_1 h_2 \ldots h_k)^{-1/2}$$

We apply the previous lemma to get

$$|S|^K \leq 2^k 8^K |I|^K (H^{-1} + \alpha\lambda^{1/2} H^{1-1/K} + |I|^{-1/2}\lambda^{-1/2} H^{-1+1/K})$$

since $\sum_{1\leq h\leq H} h^{-1/2} \leq 2H^{1/2}$, $\sum_{1\leq h\leq H} h^{1/2} \leq H^{3/2}$ and that $H_1 H_2 \ldots H_k = H^{2-2/K}$.

As in the proof of theorem 4.4, we apply again lemma 4.5 to obtain

$$|S|^K \leq 2^k 8^K |I|^K \left((\alpha^2\lambda)^{K/(4K-2)} + \alpha^{1/2}|I|^{-1/2} + |I|^{-2+1/K}\lambda^{-1/2}\right)$$

Next we take the Kth root to obtain the conclusion. ∎

5 The method of exponent pairs

As before, we want to give upper bounds for

$$S = \sum_{n=A}^{B} e(f(n))$$

but this time we assume that $I = [A,B] \subseteq [N,2N]$, for some positive integer N, and we want to provide our bounds in terms of N rather than $|I|$.

Also, we want to work with a nice family of functions f such that $f^{(j)}(x) \approx yN^{-s-j+1}$, for some $y > 0$ and $s > 0$, where the implied constants depends only on j. If $L = yn^{-s}$, we would like to find an upper bound of the form

$$|S| \ll L^k N^l$$

In this case, we say that (k,l) is an exponent pair. Clearly $(0,1)$ is an exponent pair. We will shortly provide all precise definitions.

Our method is composed of two steps. The A process consists of starting with an exponent pair (k,l) and showing that

$$A(k,l) = \left(\frac{k}{2k+2}, \frac{k+l+1}{2k+2}\right)$$

is also an exponent pair. In the B process, we show that if (k,l) is an exponent pair, then so is

$$B(k,l) = (l-1/2, k+1/2)\ .$$

Clearly $B^2(k,l) = (k,l)$. The method consists of deriving new exponent pairs of the form $A^{q_1}BA^{q_2}B\ldots A^{q_k}B(0,1)$ or $BA^{q_1}BA^{q_2}B\ldots A^{q_k}B(0,1)$ and use them to bound exponential sums.

We begin by describing our family of functions.

Definition Let N, P, y, s, ϵ be positive numbers with $\epsilon < 1/2$. We define $\mathbf{F}(N,P,s,y,\epsilon)$ to be the set of functions f with P continuous derivatives on I such that for all $0 \leq p \leq P-1$ and $A \leq x \leq B$

$$\left|f^{(p+1)}(x) - (-1)^p (s)_p y x^{-s-p}\right| \leq \epsilon (s)_p y x^{-s-p} \tag{16}$$

where $(s)_0 = 1$ and $(s)_p = s(s+1)\ldots(s+p-1)$ for $p \geq 1$.

Remark Let

$$F(x) = \begin{cases} \frac{yx^{1-s}}{1-s} & \text{if } s \neq 1 \\ y\log x & \text{if } s = 1 \end{cases} \tag{17}$$

Then (16) can be rewritten as

$$\left|f^{(p+1)}(x) - F^{(p+1)}(x)\right| \leq \epsilon\left|F^{(p+1)}(x)\right| \tag{18}$$

To begin with, it is clear that $F(x) \in \mathbf{F}(N,P,s,y,\epsilon)$, for all $\epsilon > 0$, $P > 0$. In some sense, $\mathbf{F}(N,P,s,y,\epsilon)$ consists of the functions which are "close" from $F(x)$.

Definition Let k,l be such that $0 \le k \le 1/2 \le l \le 1$. Suppose that for every $s > 0$, there exists $P = P(k,l,s)$ and $\epsilon = \epsilon(k,l,s) < 1/2$ such that for all $N > 0$, $y > 0$ and all $f \in \mathbf{F}(N,P,s,y,\epsilon)$, we have that

$$|S| \ll_{k,l,s} (yN^{-s})^k N^l + y^{-1}N^s \tag{19}$$

Then we say that (k,l) is an *exponent pair.*

Remark When proving (k,l) exponent pair, we claim that we may assume $yN^{-s} \ge 1$.

- If $yN^{-s} \le 1/2$, then $0 < \frac{1}{2}y(2N)^{-s} \le f'(x) \le \frac{3}{2}yN^{-s} < 1$ and we simply apply apply theorem 4.1 to obtain $|S| \ll_s y^{-1}N^s$
- If $1/2 \le yN^{-s} < 1$, then we apply theorem 4.2 to obtain

  $$|S| \ll N(syN^{-s-1})^{1/2} + (syN^{-s-1})^{-1/2} \ll_s N^{1/2} \ll (yN^{-s})^k N^l$$

  since $l \ge 1/2$.

Remark Note that we can show $B(0,1) = (1/2,1/2)$ is an exponent pair by simply applying theorem 4.2. However we need more advanced methods in order to derive other exponent pairs.

Before we proceed with the proofs of the A and B processes, we state a very useful application of the exponent pairs.

Lemma 5.1 *Say (k,l) is an exponent pair and let P and ϵ be the corresponding parameters given by the definition of exponent pairs. If $f \in \mathbf{F}(N,P,s,y,\epsilon)$, then*

$$\left|\sum_{n\in I}\psi(f(n))\right| \ll y^{\frac{k}{k+1}} N^{\frac{(1-s)k+l}{k+1}} + y^{-1}N^s$$

Let $J \ge 1$ a parameter to be chosen later.

Using the Fourier series expansion, we see that if x is not an integer

$$\psi(x) = -\frac{1}{2\pi i}\sum_{j\ne 0}\frac{e(jx)}{j} = -\sum_{j>1}\frac{1}{\pi j}\sin(2\pi jx)$$

According to [Vaa85] or [GK91, Appendix], there exists coefficients $|a(j)| \le \left|j^{-1}\right|$ such that

$$\psi(x) \ll J^{-1} + \sum_{1\le|j|\le J} a(j)e(jx)$$

Hence

$$\left|\sum_{n\in I}\psi(f(n))\right| \ll NJ^{-1} + \sum_{1\le j\le J}\frac{1}{j}\left|\sum_{n\in I}e(jf(n))\right| \tag{20}$$

By the definition of the exponent pair (k,l) applied to the inner sum, we obtain

$$\begin{aligned}\left|\sum_{n\in I}\psi(f(n))\right| &\ll NJ^{-1} + \sum_{1\le j\le J} j^{-1}\left((jyN^{-s})^k + j^{-1}y^{-1}N^s\right)\\ &\ll NJ^{-1} + J^kN^{l-ks}y^k + y^{-1}N^s\end{aligned}$$

We want to choose $J \ge 1$ such that our estimate is minimised. If $y^{-1}N^s < 1$, then similarly to remark after the definition of the exponent pairs, the inner sum in (20) is bounded above by $j^{-1}y^{-1}N^s$, so just by taking J large enough we obtain that$\left|\sum_{n\in I}\psi(f(n))\right| \ll y^{-1}N^s$.

Otherwise, we choose J such that $J^{k+1} = y^{-k}N^{sk-l+1}$, which implies the conclusion. ∎

From now on, most of our inequalities will be of the type $A \ll_s B$ and we will omit the s index for notational convenience. Most of this section are inspired by Chapters 3 and 4 from [GK91].

5.1 A-process

For $h \leq B-A$, define $f_1 : [A, B-h] \to \mathbb{R}$ by $f_1(x) = f(x) - f(x+h)$. We would like to show that if f is well behaved, then so is f_1.

Lemma 5.2 *Let $f \in \mathbf{F}(N, P, s, y, \epsilon)$ and $1 \leq h \leq \min(B-A, 2\epsilon N/(s+P))$. Then $f_1 \in \mathbf{F}(N, P-1, s+1, shy, 3\epsilon)$.*

Proof Let $G(x) = -hyx^{-s}$. The conclusion is equivalent to showing that

$$\left|f_1^{(p+1)}(x) - G^{(p+1)}(x)\right| \leq 3\epsilon\left|G^{(p+1)}(x)\right|$$

Let $F(x)$ be as in (17) and $F_1(x) = F(x) - F(x+h)$. We see that for $0 \leq p \leq P-2$ and $A \leq x \leq B-h$ we have that

$$\begin{aligned}\left|f_1^{(p+1)}(x) - F_1^{(p+1)}(x)\right| &= \left|\int_x^{x+h} f^{(p+2)}(u) - F^{(p+2)}(u)\,\mathrm{d}u\right| \\ &\leq \epsilon \int_x^{x+h} \left|F^{(p+2)}(u)\right| \mathrm{d}u = \epsilon\left|F_1^{(p+1)}(x)\right|\end{aligned}$$

where we are using (18) and that $F^{(p+2)}$ has constant sign.

Then we see that

$$\begin{aligned}\left|F_1^{(p+1)}(x) - G^{(p+1)}(x)\right| &= y(s)_{p+1}\left|\int_x^{x+h} (u^{-s-p-1} - x^{-s-p-1})\,\mathrm{d}u\right| \\ &= y(s)_{p+2}\left|\int_x^{x+h}\int_x^{u} w^{-s-p-2}\,\mathrm{d}w\,\mathrm{d}u\right| \\ &\leq \frac{1}{2}h^2 y(s)_{p+2}\, x^{-s-p-2} \\ &\leq \epsilon(s)_{p+1} hyx^{-s-p-1} = \epsilon\left|G^{(p+1)}(x)\right|\end{aligned}$$

where the last inequality follows from $h \leq 2\epsilon N/(s+P)$. This implies that

$$\left|F_1^{(p+1)}(x) - G^{(p+1)}(x)\right| \leq (1+\epsilon)\left|G^{(p+1)}(x)\right|$$

and combining this with the first inequality that we found we have

$$\left|f_1^{(p+1)}(x) - G^{(p+1)}(x)\right| \leq (2\epsilon + \epsilon^2)\left|G^{(p+1)}(x)\right|$$

and the conclusion follows since $\epsilon < 1$. ∎

Theorem 5.3 *If (k, l) is an exponent pair, then*

$$(\kappa, \lambda) = \left(\frac{k}{2k+2}, \frac{k+l+1}{2k+2}\right)$$

is an exponent pair.

Proof First, we notice that

$$0 \leq \kappa = \frac{k}{2k+2} \leq \frac{1}{2} \text{ and } \frac{1}{2} \leq \frac{1}{2} + \frac{l}{2k+2} \leq 1$$

since $0 \leq k \leq 1/2$ and $1/2 \leq l \leq 1$.

Let y, N, s be positive. We need to show there exists $P' > 0$ and $0 < \epsilon' < 1/2$ such that if $f \in \mathbf{F}(N, P', s, y, \epsilon')$, then

$$\sum_{n \in I} e(f(n)) \ll (yN^{-s})^{\kappa} N^{\lambda} + y^{-1}N^{s}$$

Also, by an earlier remark, we may assume $L = yN^{-s} \geq 1$.

Since (k, l), we now there exists P and ϵ such that if $f \in \mathbf{F}(N, P, s, y, \epsilon)$, then

$$\sum_{n \in I} e(f(n)) \ll (yN^{-s})^k N^l + y^{-1} N^s$$

We take $P' = P + 1$ and $\epsilon' = \epsilon/3$. Let $f \in \mathbf{F}(N, P', s, y, \epsilon')$.

Recall that $S_1(h) = \sum_{n=A}^{B-h} e(f_1(n))$ and since $f_1 \in \mathbf{F}(N, P, s+1, shy, \epsilon)$ (using the previous lemma) we get that

$$|S_1(h)| \ll (shyN^{-s-1})^k N^l + (shy)^{-1} N^{s+1}$$

Let $H \leq \min(B - A, 2\epsilon N/(s + P))$. Using (11) we get

$$\begin{aligned} |S|^2 &\ll \frac{N^2}{H} + \frac{N}{H} \sum_{1 \leq h \leq H} |S_1(h)| \\ &\ll H^{-1}N^2 + H^{-1}N \sum_{1 \leq h \leq H} \left(h^k L^k N^{l-k} + h^{-1} L^{-1} N \right) \\ &\ll H^{-1}N^2 + H^k L^k N^{l-k+1} + H^{-1} L^{-1} N^2 \log N \end{aligned}$$

If $1 \leq L \leq \log N$, then using theorem 4.2, we obtain

$$S \ll N(yN^{-s-1})^{1/2} + (yN^{-s-1})^{-1/2} \ll N^{1/2} L^{1/2} \ll N^{2/3} \ll L^\kappa N^\lambda$$

since $\lambda = 1/2 + l/(2k + 2) \geq 2/3$.

If $L \geq \log N$, then the equation above gives us $|S|^2 \ll H^{-1}N^2 + H^k L^k N^{l-k+1}$. We apply again lemma 4.5 using $0 < H < (B - A)$ to get

$$|S|^2 \ll \left(L^k N^{l-k+1} N^{2k}\right)^{1/(k+1)} + N(B - A)^{-1}$$

Hence

$$|S| \ll L^\kappa N^\lambda + N(B - A)^{-1/2} \tag{21}$$

If the first term dominates, we are done, otherwise

$$S \ll \min(N(B - A)^{-1/2}, (B - A)) \ll N^{2/3}$$

which is enough as we saw above. ∎

5.2 B-process

Recall that we have $f : [A, B] \to \mathbb{R}$, where $N \leq A \leq B \leq 2N$ and we assume $f \in \mathbf{F}(N, P, s, y, \epsilon)$, in particular f' is decreasing. Let $\alpha = f'(B)$ and $\beta = f'(A)$ and $g : [\alpha, \beta] \to [A, B]$ be the inverse of f', so $f'(g(x)) = x$. Let

$$\phi : [\alpha, \beta] \to \mathbb{R}, \quad x \mapsto xg(x) - f(g(x))$$

so we have that $\phi'(x) = g(x)$. We will show that ϕ is also "nice".

Lemma 5.4 *Let f and ϕ be as above. Let $\sigma = 1/s$ and $\mu = y^\sigma$. Then there exists $C = C(s, P)$ such that for any J with $\alpha \leq J \leq \beta$, then the restriction of ϕ to $[\alpha, \beta] \cap [J, 2J]$ is in $\mathbf{F}(J, P, \sigma, \mu, C\epsilon)$.*

Proof Let F be as in (17) and let G be the inverse of F' (i.e. $F'(G(x)) = x$). Since $F'(x) = yx^{-s}$ it is easy to check that $G(x) = \mu x^{-\sigma}$. Define

$$\Phi(x) := xG(x) - F(G(x))$$

Then $\Phi'(x) = G(x) = \mu x^{-\sigma}$. So it suffices to show there exists constant C such that for all $0 \leq p \leq P - 1$ and $\alpha \leq x \leq \beta$

$$\left| \phi^{(p+1)}(x) - \Phi^{(p+1)}(x) \right| < C\epsilon \Phi^{(p+1)}(x) \ .$$

We know that $f'(\phi'(x)) = x$, so be differentiating both sides we get

$$\phi"(x) = \frac{1}{f"(g(x))}$$

Differentiating again and using that $g'(x) = 1/(f"(g(x)))$ we obtain that

$$\phi^{(3)}(x) = -\frac{f^{(3)}(g(x))}{(f"(g(x)))^3}$$

Hence an straightforward induction argument gives us that for $p \geq 1$ we have

$$\phi^{(p+1)}(x) = \frac{1}{((f"(g(x)))^{2p-1}} \sum_{\substack{u_1+\cdots+u_{p-1}=3p-3 \\ 2\leq u_i \leq p+1}} w(u_1, \ldots, u_{p-1}) f^{(u_1)}(g(x)) \ldots f^{(u_{p-1})}(g(x))$$

for some constants $w(u_1, \ldots, u_{p-1})$, Similarly we obtain

$$\Phi^{(p+1)}(x) = \frac{1}{((F"(G(x)))^{2p-1}} \sum_{\substack{u_1+\cdots+u_{p-1}=3p-3 \\ 2\leq u_i \leq p+1}} w(u_1, \ldots, u_{p-1}) F^{(u_1)}(G(x)) \ldots F^{(u_{p-1})}(G(x))$$

Recall (18) which gives us

$$|f'(g(x)) - F'(g(x))| = |x - y(g(x))^{-s}| \leq \epsilon y g(x)^{-s}$$

Hence

$$(1-\epsilon)^{1/s} y^{1/s} x^{-1/s} \leq g(x) \leq (1+\epsilon)^{1/s} y^{1/s} x^{-1/s}$$

which can be rewritten as $(1-\epsilon)^{\sigma} G(x) \leq g(x) \leq (1-\epsilon)^{\sigma} G(x)$, which in turn implies

$$|g(x) - G(x)| \ll_s \epsilon G(x)$$

Also, we have that for all $0 \leq p < P-1$

$$|f^{(p+1)}(g(x)) - F^{(p+1)}(g(x))| \leq \epsilon |F^{(p+1)}(g(x))| \ll_{s,P} \epsilon y N^{-s-p}$$

Applying the mean value theorem, there exists some t in the interval with endpoints $g(x)$ and $G(x)$ such that

$$|F^{(p+1)}(g(x)) - F^{(p+1)}(G(x))| = |F^{(p+2)}(t)(G(x) - g(x))| \ll_{s,P} \epsilon y N^{-s-p-1} G(x) \ll_{s,P} \epsilon y N^{-s-p}$$

Hence

$$|f^{(p+1)}(g(x)) - F^{(p+1)}(G(x))| \ll_{s,P} \epsilon y N^{-s-p}$$

Now we use our expressions for $\phi^{(p+1)}(x)$ and $\Phi^{(p+1)}(x)$ and to make our notation easier, let $z = G(x)$, to get:

$$\begin{aligned}
|\phi^{(p+1)}(x) - \Phi^{(p+1)}(x)| &\ll_{s,P} \frac{\epsilon}{|F"(z)|^{2p-1}} \sum_{\substack{u_1+\cdots+u_{p-1}=3p-3 \\ 2\leq u_i \leq p+1}} (yz^{-s-u_1+1}) \ldots (yz^{-s-u_{p-1}+1}) \\
&\ll_{s,P} \frac{\epsilon}{(yz^{-s-1})^{2p-1}} y^{p-1} z^{-s(p-1)-2p+2} \\
&\ll_{s,P} \epsilon y^{-p} z^{ps+1} \ll_{s,P} \ll_{s,P} \epsilon \Phi^{(p+1)}(x) \quad \blacksquare
\end{aligned}$$

Before we proceed with the proof of the B-process, we need the following lemma:

Lemma 5.5 *Say f has 4 continuous derivatives and $f" < 0$. Also, suppose that there exists some $Q > 0$ such that*

$$f"(x) \approx QN^{-2}, \quad f^{(3)} \ll QN^{-3}, \quad f^{(4)}(x) \ll QN^{-4} \ .$$

With the same notation as above, we have

$$S = \sum_{\alpha \leq x \leq \beta} \frac{e(-\phi(x) - 1/8)}{|f"(g(x))|^{1/2}} + O\left(\log(QN^{-1}+2) + Q^{-1/2}N\right) \ .$$

Due to constraints on the size of this project, we omit the proof. Proofs can be found in [GK91, Lemma 3.6] or [Tit86, Lemma 4.6].

Theorem 5.6 *If* (k, l) *is an exponent pair, then*

$$(\kappa, \lambda) = (l - 1/2, k + 1/2)$$

is an exponent pair.

Proof First, we notice that since $0 \leq k \leq 1/2 \leq l \leq 1$, we indeed have that $0 \leq \kappa \leq 1/2 \leq \lambda \leq 1$. If

Let $L = yN^{-s}$ and we may assume $L \geq 1$. We know there exists $P > 0$ and $\epsilon > 0$ corresponding to the exponent pair (k, l). Let $P' = P$, $\epsilon' = \epsilon/C$ (where C comes from lemma 5.4). Let $f \in \mathbf{F}(N, P', s, y, \epsilon')$. We want to show that $\sum_{n \in I} e(f(n)) \ll L^\kappa N^\lambda$.

Since f clearly satisfies the hypothesis lemma 5.5 with $Q = LN$, hence

$$S = \sum_{\alpha \leq x \leq \beta} \frac{e(-\phi(x) - 1/8)}{|f"(g(x))|^{1/2}} + O\left(\log(2L) + L^{-1/2}N^{1/2}\right)$$

Denote by S' the sum from the right hand side.

Now we apply lemma 5.4 and use that (k, l) is an exponent pair and $\phi \in \mathbf{F}(J, P, \sigma, \mu, \epsilon)$ to get

$$T(w) := \sum_{\alpha \leq x \leq w} (e(\phi(x)) \ll (\mu J^{-\sigma})^k J^l + \mu^{-1} J^\sigma \ll N^k J^l + N^{-1}$$

Putting the last 2 equations together we have that

$$\begin{aligned}
|S'| &\ll \left[\overline{T(w)}|f"(g(x))|^{-1/2}\right]_\alpha^\beta + \int_\alpha^\beta \left|T(x)\frac{\mathrm{d}}{\mathrm{d}x}|f"(g(x))|^{-1/2}\right| \mathrm{d}x \\
&\ll \left(N^k L^l + N^{-1}\right)\left((LN^{-1})^{-1/2} + \int_\alpha^\beta \left|\frac{\mathrm{d}}{\mathrm{d}x}|f"(g(x))|^{-1/2}\right| \mathrm{d}x\right) \\
&\ll L^\kappa N^\lambda + L^{-1/2}N^{-1/2}
\end{aligned}$$

So putting everything together, we have that

$$|S| \ll L^\kappa N^\lambda + \log(2L) + L^{-1/2}N^{1/2}$$

Now, if we assume $\kappa > 0$, we have that the first term dominates (since we assume $L \geq 1$) and we are done.

If $\kappa = 0$, then $l = 1/2$. In [GK91, Section 3.3] it is proved that the only exponent pair of the type $(k, 1/2)$ is $(1/2, 1/2)$. This means that $(\kappa, \lambda) = (0, 1)$, which is clearly an exponent pair.

5.3 Applying exponent pairs

Recall that our goal was to estimate the error term in the Gauss circle problem.

Recall that

$$G(x) = \sum_{n \leq x} r(n) = 4 \sum_{n \leq x} \sum_{d | n} \chi(d) = 4 \sum_{d \leq x} \chi(d) \left\lfloor \frac{x}{d} \right\rfloor \tag{22}$$

Define

$$S(x) = \sum_{n \leq x} \chi(n) \tag{23}$$

Then an easy case by case analysis implies that

$$S(x) = \frac{1}{2} - \psi\left(\frac{x-1}{4}\right) + \psi\left(\frac{x-3}{4}\right) \tag{24}$$

Lemma 5.7

$$\sum_{d\le x}\frac{\chi(d)}{d}=\frac{\pi}{4}+\frac{S(x)-1/2}{x}+O\left(\frac{1}{x^2}\right)$$

Proof

$$\sum_{d\le x}\frac{\chi(d)}{d}=\frac{\pi}{4}-\sum_{d>x}\frac{\chi(d)}{d}=\frac{\pi}{4}-\sum_{d>x}\frac{S(d)-S(d-1)}{d}=\frac{\pi}{4}-\frac{S(x)}{x+1}+\sum_{d>x}S(d)\left(\frac{1}{d}-\frac{1}{d+1}\right)$$
$$=\frac{\pi}{4}-\frac{S(x)}{x}+\int_x^\infty\frac{S(u)}{u^2}\,\mathrm{d}u+O\left(\frac{1}{x^2}\right)$$

But now

$$\int_x^\infty\psi\left(\frac{u-a}{4}\right)\frac{\mathrm{d}u}{u^2}=4\int_{(x-a)/4}^\infty\psi(u)\frac{1}{(4u+a)^2}\,\mathrm{d}u\ll\frac{1}{x^2}$$

for $a=1$ or $a=3$. Putting this together with (24) gives the result. ∎

Theorem 5.8 *If* (k,l) *is an exponent pair different from* $(1/2,1/2)$*, then*

$$E(x)\ll x^{\frac{k+l}{2k+2}}\ .$$

Proof Recall from (22) that

$$G(x)=4\sum_{n\le x}\sum_{d|n}\chi(d)=4\sum_{md\le x}\chi(d)=$$
$$=4\left(\sum_{d\le\sqrt{x}}\sum_{m\le x/d}\chi(d)+\sum_{m\le\sqrt{x}}\sum_{d\le x/m}\chi(d)-\sum_{d\le\sqrt{x}}\sum_{m\le\sqrt{x}}\chi(d)\right)=$$
$$=4\left(\sum_{d\le\sqrt{x}}\chi(d)\left\lfloor\frac{x}{d}\right\rfloor+\sum_{m\le\sqrt{x}}S\left(\frac{x}{m}\right)-\lfloor\sqrt{x}\rfloor S(\sqrt{x})\right)$$

We evaluate each of the three sums individually. Using the previous lemma, the first sum is

$$S_1=\sum_{d\le\sqrt{x}}\chi(d)\left\lfloor\frac{x}{d}\right\rfloor=\sum_{d\le\sqrt{x}}\chi(d)\left(\frac{x}{d}-\frac{1}{2}-\psi\left(\frac{x}{d}\right)\right)$$
$$=x\sum_{d\le\sqrt{x}}\frac{\chi(d)}{d}-\frac{1}{2}S\left(\sqrt{x}\right)-\sum_{d\le\sqrt{x}}\chi(d)\psi\left(\frac{x}{d}\right)$$
$$=\frac{\pi x}{4}+\sqrt{x}(S(\sqrt{x})-1/2)-\sum_{d\le\sqrt{x}}\chi(d)\psi\left(\frac{x}{d}\right)+O(1)$$

The second sum is

$$S_2=\sum_{m\le\sqrt{x}}S\left(\frac{x}{m}\right)=\frac{1}{2}\sqrt{x}-\sum_{m\le\sqrt{x}}\left(\psi\left(\frac{x-m}{4m}\right)-\psi\left(\frac{x-3m}{4m}\right)\right)+O(1)$$

Putting everything together we get

$$G(x)=\pi x+4\sum_{d\le\sqrt{x}/4}\left(\psi\left(\frac{x}{4d+3}\right)-\psi\left(\frac{x}{4d+1}\right)+\psi\left(\frac{x}{4d}-\frac{3}{4}\right)-\psi\left(\frac{x}{4d}-\frac{1}{4}\right)\right)+O(1)$$

Let $f(y)=-x/(4y)$ considered in the interval $[1,\sqrt{x}]$. Then $f\in\mathbf{F}(N,P,2,x/4,\epsilon)$ for all $N\le\sqrt{x}/2$. Hence we can apply lemma 5.1. We consider intervals of the form $I_j=\{n:2^{-j}\sqrt{x}<n\le 2^{-j+1}\sqrt{x}\}$.

Hence

$$\left|\sum_{d\leq\sqrt{x}/4}\psi\left(-\frac{x}{4d}\right)\right|\leq\sum_{3\leq j\leq J}\left|\sum_{n\in I_j}\psi\left(-\frac{x}{4d}\right)\right|$$
$$\leq\sum_{3\leq j\leq J}\left((x/4)^{\frac{k}{k+1}}(2^{-j}x^{1/2})^{\frac{l-k}{k+1}}+2^{2-2j}\right)$$
$$\ll x^{\frac{k+l}{2k+2}}\sum_{3\leq j\leq J}2^{-\frac{j(l-k)}{k+1}}+\sum_{3\leq j\leq J}2^{-2j}$$

The second sum converges, and if $k<l$, the first sum also converges.

If we add a constant to our function f or we consider the slightly different version $f(y)=-x/(4y+1)$ we obtain the same estimate. The result follows from noticing that if x is not an integer, then $\psi(x)=-\psi(-x)$. ∎

Now take

$$(k,l)=BA^3B(0,1)=\left(\frac{11}{30},\frac{26}{30}\right)\ .$$

Then we have that $E(x)\ll x^{27/82}$, where $27/82=0.3292....$

6 Obtaining an explicit constant

We follow the approach from [Jam]. We want to use bound $E(r)=G(r)-\pi r$ by an exponential sum and use theorem 4.2.

Let

$$N(r)=\#\{(m,n)\in\mathbb{Z}^2\mid m^2+n^2\leq r,m>0,n\geq 0\}$$

so clearly we have that $G(r)=1+4N(r)$ (we can easily see that $N(r)$ counts the number of integer lattice points in a quadrant, excluding the origin).

We begin by estimating $N(r)$. Let $M=\left\lfloor\sqrt{r/2}\right\rfloor$. Then we have that

$$N(r)=2\sum_{m=1}^{M}\lfloor\sqrt{r-m^2}-m\rfloor+M+\lfloor\sqrt{r}\rfloor$$

The sum counts the pair of points $\{(m,n)\in\mathbb{Z}^2\mid m^2+n^2\leq r,m>0,n>0,m\neq n\}$, the second terms comes from the terms with $m=n$ and the last one from the terms with $n=0$.

Let $f(x)=\sqrt{r-x^2}-x$. Hence we have

$$N(r)=2\sum_{m=1}^{M}\lfloor f(m)\rfloor+M+\lfloor\sqrt{r}\rfloor=$$
$$=2\sum_{m=1}^{M}(f(m)-\psi(f(m))-\frac{1}{2})+M+\lfloor\sqrt{r}\rfloor=$$
$$=2\sum_{m=1}^{M}f(m)-2\sum_{m=1}^{M}\psi(f(m))+\lfloor\sqrt{r}\rfloor$$

We begin by evaluating the first sum. We expect to be well approximated by

$$\int_0^{\sqrt{r/2}}f(x)=\pi r/8$$

(the area of an octant of the circle).

Indeed, we use the trapezium rule:

$$\begin{aligned}\int_0^M f'(x)\psi(x)\,\mathrm{d}x &= \sum_{m=0}^{M-1}\int_0^1 f'(x+m)\left(x-\frac{1}{2}\right)\mathrm{d}x = \\ &= \sum_{m=0}^{M-1}\left(\frac{1}{2}f(x+m+1)+\frac{1}{2}f(x+m)-\int_0^1 f(x+m)\,\mathrm{d}x\right) = \\ &= \frac{1}{2}f(0)+\sum_{m=0}^{M-1}f(m)+\frac{1}{2}f(M)-\int_0^M f(x)\,\mathrm{d}x\end{aligned}$$

Hence $\int_0^M f'(x)\psi(x)\,\mathrm{d}x$ is the error in approximating $\int_0^M f(x)\,\mathrm{d}x$ using strips of unit width.

Also,

$$\begin{aligned}\left|\int_0^M f'(x)\psi(x)\,\mathrm{d}x\right| &\le \sum_{m=0}^{M-1}\left|\int_0^1 f'(x+m)\left(x-\frac{1}{2}\right)\mathrm{d}x\right| \\ &= \sum_{m=0}^{M-1}\left|\int_0^1 f''(x+m)\left(\frac{1}{2}x^2-\frac{1}{2}x\right)\mathrm{d}x\right| \\ &\le \frac{1}{8}\int_0^M |f''(x)|\,\mathrm{d}x\end{aligned}$$

Since $f'(x) = -x(r-x^2)^{-1/2}-1$ we notice that $f''(x)\le 0$ and $f'(x)\in[-2,-1]$ for $x\in[0,r/\sqrt{2}]$, hence

$$\left|\int_0^M f'(x)\psi(x)\,\mathrm{d}x\right| \le -\frac{1}{8}\int f''(x)\,\mathrm{d}x = \frac{1}{8}(f'(0)-f'(M)) \le \frac{1}{2}$$

Now we observe that $f(0)=r$, $f(\sqrt{r/2})=0$ and putting everything together

$$\begin{aligned}N(r) &= 2\sum_{m=1}^{M}f(m)-2\sum_{m=1}^{M}\psi(f(m))+\lfloor\sqrt{r}\rfloor \\ &= 2\int_0^M f(x)\,\mathrm{d}x+2\int_0^M f'(x)\psi(x)\,\mathrm{d}x-f(0)+f(M)+2\sum_{m=1}^{M}\psi(f(m))+\lfloor\sqrt{r}\rfloor \\ &= \pi r/4-2\sum_{m=1}^{M}\psi(f(m))+C(r)\end{aligned}$$

where $|C(r)|\le 8$ for all r. Here we are using that

$$\int_M^{\sqrt{r/2}} f(x)\,\mathrm{d}x \le 2$$

and $|f(M)|\le 2$ since $f'(x)\in[-2,-1]$ and $f(\sqrt{r/2})=0$.

Define $P(r)=N(r)-\pi r/4$. Our goal is to find an upper bound for $P(r)$.

We want to force an averaging argument. We begin by observing that since $N(X+y)\ge N(X)$, it follows that

$$P(X)\le P(X+y)+\pi y/4\ ,$$

for all $X,y\ge 0$. Integrating both sides with y ranging from 0 to Y, for some $Y>0$, we obtain

$$P(X)\le\frac{1}{Y}\int_0^Y P(X+y)\,\mathrm{d}y+\frac{\pi Y}{8}\ .$$

Similarly

$$P(X)\ge\frac{1}{Y}\int_0^Y P(X-y)\,\mathrm{d}y-\frac{\pi Y}{8}\ .$$

Putting everything together, we obtain

$$|P(X)| \leq \frac{1}{Y} \max\left(\left| \int_X^{X+Y} P(x)\,\mathrm{d}x \right|, \left| \int_{X-Y}^{X} P(x)\,\mathrm{d}x \right| \right) + \frac{\pi Y}{8} \ . \tag{25}$$

In our applications Y will be small compared to X, so $P(X)$ is bounded by its average on a short interval. We will choose in particular $Y < \sqrt{X}$.

Recall that

$$P(X) = -2 \sum_{m=1}^{\sqrt{X/2}} \psi(\sqrt{X - m^2}) + C(X) \ . \tag{26}$$

Therefore

$$\int_X^{X+Y} P(x)\,\mathrm{d}x = -2 \int_X^{X+Y} \sum_{m=1}^{\sqrt{x/2}} \psi(\sqrt{x - m^2})\,\mathrm{d}x + \int_X^{X+Y} C(x)\,\mathrm{d}x = -2 \sum_{m=1}^{\sqrt{X/2}} \int_X^{X+Y} \psi(\sqrt{x - m^2})\,\mathrm{d}x + O(Y) \tag{27}$$

where the implicit constant implied in $O(Y)$ is at most 9 (since $\sqrt{(X+Y)/2} \leq \sqrt{X/2} + 1$ and $|\psi|$ bounded by 1/2).

So our goal is to evaluate the integral from inside the sum. Using change of variables we obtain

$$\int_X^{X+Y} \psi(\sqrt{x - m^2})\,\mathrm{d}x = 2 \int_{\sqrt{X - m^2}}^{\sqrt{X+Y-m^2}} \psi(x) x\,\mathrm{d}x \tag{28}$$

If N is a positive integer, then

$$\int_0^N \psi(x) x\,\mathrm{d}x = \sum_{n=0}^{N-1} \int_0^1 (x - 1/2)(x + n)\,\mathrm{d}x = \sum_{n=0}^{N-1} \int_0^1 (x^2 - \frac{1}{2}x)\,\mathrm{d}x = N/12$$

Also, for a real number a,

$$\int_{\lfloor a \rfloor}^a \psi(x) x\,\mathrm{d}x = \int_0^{\{a\}} (x - 1/2)(\lfloor a \rfloor + x)\,\mathrm{d}x = \lfloor a \rfloor \left(\frac{1}{2}\{a\}^2 - \frac{1}{2}\{a\} \right) + \frac{\{a\}^3}{3} - \frac{\{a\}^2}{4}$$

Hence

$$\int_0^a \psi(x) x\,\mathrm{d}x = \frac{\lfloor a \rfloor}{2} \left(\{a\}^2 - \{a\} + \frac{1}{6} \right) + \frac{\{a\}^3}{3} - \frac{\{a\}^2}{4} \tag{29}$$

Since we recognise the second Bernoulli polynomial in $\{a\}$, it makes sense to define the periodified Bernoulli polynomial

$$B_2(x) := B_2(\{x\}) = \{x\}^2 - \{x\} + \frac{1}{6}$$

which is a periodic function function with period 1, and has Fourier series expansion

$$B_2(x) = \frac{1}{2\pi^2} \sum_{n \neq 0} \frac{e(nx)}{n^2} = \frac{1}{\pi^2} \operatorname{Re}\left\{ \sum_{n>0} \frac{e(nx)}{n^2} \right\} . \tag{30}$$

Hence

$$\int_0^a \psi(x) x\,\mathrm{d}x = \frac{a}{2} B_2(a) + \frac{5\{a\}^3}{6} - \frac{3\{a\}^2}{4} + \frac{\{a\}}{12} = \frac{a}{2} B_2(a) + r(a) \tag{31}$$

where $|r(a)| \leq 1/6$, for all positive reals a. Hence from (28) and (31) we obtain

$$\int_X^{X+Y} \psi(\sqrt{x - m^2})\,\mathrm{d}x = \sqrt{X + Y - m^2} B_2(\sqrt{X + Y - m^2}) - \sqrt{X - m^2} B_2(\sqrt{X - m^2}) + O(1)$$

where the implicit constant is at most 4/6 in absolute value. Next, since we assume $Y \leq \sqrt{X}$ and $m \leq \sqrt{X/2}$, we have that $\sqrt{X + Y - m^2} \leq \sqrt{X - m^2} + 1$, and since $|B_2(x)| \leq 1/6$, for all x:

$$\left| \int_X^{X+Y} \psi(\sqrt{x - m^2})\,\mathrm{d}x - \sqrt{X - m^2} \left(B_2(\sqrt{X + Y - m^2}) - B_2(\sqrt{X - m^2}) \right) \right| \leq 1 \tag{32}$$

Putting together (27), (30), and (32), we get

$$\left|\int_X^{X+Y} P(x)\,\mathrm{d}x\right| \le \frac{2}{\pi^2}\sum_{m=1}^{\sqrt{X/2}} \sqrt{X-m^2}\sum_{n>0}\frac{1}{n^2}\,\mathrm{Re}\Big\{e(n\sqrt{X-m^2}) - e(n\sqrt{X+Y-m^2})\Big\} + E(X,Y)$$

$$\le \frac{2}{\pi^2}\sum_{n>0}\frac{1}{n^2}\,\mathrm{Re}\sum_{m=1}^{\sqrt{X/2}} \sqrt{X-m^2}\,\Big(e(n\sqrt{X-m^2}) - e(n\sqrt{X+Y-m^2})\Big) + E(X,Y) \tag{33}$$

where $|E(X,Y)| \le \frac{\sqrt{2X}}{\pi^2} + 9Y \le \frac{1}{6}X^{1/2} + 9Y$.

We need the following partial summation lemma:

Lemma 6.1 *Let $A < B$ be integers and $f : [A,B] \to \mathbb{R}$ continuously differentiable and $c_n \in \mathbb{R}$, for all $n \in [A,B]$ integer. Then*

$$\left|\sum_{n=A}^{B} g(n)c_n\right| \le \left||g(B)| + \int_A^B |g'(x)|\,\mathrm{d}x\right| \max_{A\le n\le B}\left|\sum_{m=A}^{n} c_m\right| .$$

Proof For $x \in [A,B]$, define $F(x) := \sum_{A\le n\le x} c_n$. Then

$$\begin{aligned}\sum_{n=A}^{B} g(n)c_n &= g(A)F(A) + \sum_{n=A+1}^{B} g(n)(F(n)-F(n-1))\\ &= \sum_{n=A}^{B-1} F(n)(g(n)-g(n+1)) + g(B)F(B)\\ &= -\int_A^B F(x)g'(x) + g(B)F(B)\end{aligned}$$

since $F(x)$ is constant on intervals of the form $[n, n+1)$, n integer. Hence

$$\left|\sum_{n=A}^{B} g(n)c_n\right| \le \max_{A\le n\le B} F(n)\cdot\left||g(B)| + \int_A^B |g'(x)|\,\mathrm{d}x\right| . \quad \blacksquare$$

Now we are aready to put all estimates together. Let $f_n(x) = n\sqrt{X-x^2}$. Then $f_n'(x) = -nx(X-x^2)^{-1/2}$ and $f_n''(x) = -nX(X-x^2)^{-3/2}$. Hence for $x \in [0, \sqrt{X/2}]$, we have that

$$-\sqrt{8}nX^{-1/2} \le f_n''(x) \le -nX^{-1/2} .$$

We apply Theorem 4.2 to obtain that, for all $t \le \sqrt{X/2}$

$$\left|\sum_{m=1}^{t} e(n\sqrt{X-m^2})\right| \le 8X^{1/4}n^{1/2} + 8X^{1/4}n^{-1/2} \le 16X^{1/4}n^{1/2} \tag{34}$$

Hence applying the partial summation lemma, we obtain

$$\left|\sum_{m=1}^{t} \sqrt{X-m^2}\,e(n\sqrt{X-m^2})\right| \le 32X^{3/4}n^{1/2} \tag{35}$$

Similarly, we obtain that

$$\left|\sum_{m=1}^{t} \sqrt{X-m^2}\,e(n\sqrt{X+Y-m^2})\right| \le 32X^{3/4}n^{1/2} \tag{36}$$

as long as $X \ge 16$ say.

Using Taylor expansion, we have that

$$\begin{aligned}
\sqrt{X+Y-m^2} &= \sqrt{X-m^2}\left(1+\frac{Y}{X-m^2}\right)^{1/2} \\
&= \sqrt{X-m^2}\left(1+\frac{1}{2}\frac{Y}{X-m^2}+O\left(\frac{Y^2}{X^2}\right)\right) \\
&= \sqrt{X-m^2}\left(1+\frac{Y}{2\sqrt{X-m^2}}\right)+O\left(\frac{Y^2}{X^{3/2}}\right)
\end{aligned}$$

Therefore

$$e(n\sqrt{X+Y-m^2}) = e(n\sqrt{X-m^2})e\left(\frac{nY}{2\sqrt{X-m^2}}\right)+O\left(\frac{nY^2}{X^{3/2}}\right)$$

where it is easy to check that the constant implied in $O\left(\frac{nY^2}{X^{3/2}}\right)$ is less than 8.

Next, we define

$$\begin{aligned}
I_n :=& \operatorname{Re}\sum_{m=1}^{\sqrt{X/2}} \sqrt{X-m^2}\left(e(n\sqrt{X+Y-m^2})-e(n\sqrt{X-m^2})\right) \\
=& \operatorname{Re}\sum_{1\le m\le\sqrt{X/2}} \sqrt{X-m^2}\ e(n\sqrt{X-m^2})\left(1-e\left(\frac{nY}{2\sqrt{X-m^2}}\right)\right)+O\left(\frac{nY^2}{X^{1/2}}\right) \\
=& \sum_{1\le m\le\sqrt{X/2}} \sqrt{X-m^2}\ e(n\sqrt{X-m^2})\left(1-\cos\left(\frac{\pi nY}{\sqrt{X-m^2}}\right)\right)+O\left(\frac{nY^2}{X^{1/2}}\right)
\end{aligned}$$

Now, on one hand, we can simply use (35) and(36) and the definition of I_n to deduce that

$$|I_n| \le 64X^{3/4}n^{1/2}$$

On the other hand, for $x \ge 0$, we have that $1-\cos x \le x$, hence applying again the partial summation lemma we obtain

$$|I_n| \le \frac{\sqrt{2}\pi nY}{X^{1/2}}\cdot 32X^{3/4}n^{1/2}+O\left(\frac{nY^2}{X^{1/2}}\right) \le 143\cdot X^{1/4}\cdot n^{3/2}\cdot Y+O\left(\frac{nY^2}{X^{1/2}}\right) \le 144\cdot X^{1/4}\cdot n^{3/2}\cdot Y$$

as long as $X \ge 16$.

In order to evaluate (33), we will split the summation over n in two parts. We set $Z = X^{1/2}/Y$.

$$\left|\sum_{n\ge1}\frac{1}{n^2}I_n\right| \le \sum_{n\le Z} 144\cdot X^{1/4}\cdot n^{-1/2}\cdot Y+\sum_{n>Z} 64\cdot X^{3/4}\cdot n^{-3/2}$$

We now use that $\sum_{n=1}^{Z} n^{-1/2} \le \int_0^Z x^{-1/2}\,\mathrm{d}x = 2Z^{1/2}$ and that $\sum_{n>Z} n^{-3/2} \le \int_0^Z x^{-3/2}\,\mathrm{d}x = 2Z^{-1/2}$. Hence

$$\begin{aligned}
\left|\sum_{n\ge1}\frac{1}{n^2}I_n\right| &\le 2\cdot 144\cdot X^{1/4}\cdot Y\cdot Z^{1/2}+2\cdot 64\cdot X^{3/4}\cdot Z^{-1/2} \\
&\le 416\cdot X^{1/2}\cdot Y^{1/2}
\end{aligned}$$

Plugging in (33), we obtain

$$\left|\int_X^{X+Y} P(x)\,\mathrm{d}x\right| \le 85X^{1/2}Y^{1/2}+E(X,Y)\ .$$

Since $|E(X,Y)| \leq \frac{1}{6}X^{1/2} + 9Y$, we set $Y = X^{1/3}$ and using that $\frac{1}{6}X^{1/2} + 9X^{1/3} \leq 4X^{2/3}$ for $X \geq 16$ we obtain

$$\left| \int_X^{X+Y} P(x)\,\mathrm{d}x \right| \leq 89X^{2/3}$$

We obtain a similar estimate for $\left| \int_{X-Y}^{X} P(x)\,\mathrm{d}x \right|$. Finally, plugging in back to (25), we obtain

$$|P(X)| \leq 90X^{1/3} \tag{37}$$

as long as $X \geq 16$.

Recalling that $N(X) = \pi X/2 + P(X)$ and that $G(X) = 1 + 4N(X)$, we obtain that $G(X) = \pi X + E(X)$, where

$$|E(X)| \leq 361X^{1/3}$$

as long as $X \geq 16$.

Remark Note that our approximations in obtaining our final constant are far from optimal. In fact, assuming X is sufficiently large, we can find our final constant to be arbitrarily close to

$$\frac{8}{\pi^2}(32\pi + 32\sqrt{2}) = 118.1695...$$

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