The Erdős distinct distances problem

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Abstract

The classical Erdős distinct distances problem asks what is the least number of distinct distances determined by N points in the plane. Erdős conjectured in 1946 that this number is at least $N^{1-o(1)}$. Despite the fact that the lower bound has been improved successively in the following years, the conjecture remained open until 2010, when Larry Guth and Nets Katz showed that the number of distinct distances is $\gtrsim N/\log N$. Their proof is considered an important breakthrough and it uses the polynomial method, a relatively new technique which turned out to have a wide range of applications.

In this report we aim to give an accessible and detailed exposition of their proof. We will highlight the algebraic geometry prerequisites and we will obtain our own constants.

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1 Introduction

The classical Erdős distinct distances problem asks what is the least number of distinct distances determined by N points in plane. In the original paper [4] from 1946, Erdős called this number $f(N)$. By considering N points equally spaced along a line, we note that $f(N) \leq N-1$. If we consider a $\sqrt{N} \times \sqrt{N}$ grid, then the square of each distance is the sum of two squares between 0 and N. We recall the famous Landau-Ramanujan theorem in analytic number theory:

Theorem 1.1 (Landau-Ramanujan) *Let $g(N)$ be the number of positive integers smaller than N that can be written as the sum of two squares. Then $g(N) = O(N/\sqrt{\log N})$.*

In this report, if $f, g : X \to \mathbb{R}^+$ are two functions with real values, by $f \lesssim g$ or $f = O(g)$ we mean that that there exists a constant $C > 0$ such that $f(x) \leq C \cdot g(x)$, for all $x \in X$. We can similarly define $f \gtrsim g$.

Therefore we obtain that

$$f(N) \leq g(2N) = O(2N/\sqrt{\log(2N)}) = O(N/\sqrt{\log N}) \ .$$

However, it is more difficult to obtain a good lower bound for $f(N)$. In his original paper, Erdős could only prove that $f(N) \geq (N - 3/4)^{1/2} - 1/2$. The proof of this fact is easy and worth exposing from an historical point of view.

Say $\mathscr{P} = \{p_1, \dots, p_N\}$ is a set of N points in plane and say that p_1 belongs to the convex hull of these points. We look at the distances $d(p_1, p_i)$, for $2 \leq i \leq N$. Say among these distances there are k different distances and r is the most number of times a distance occurs. Then clearly $kr \geq N-1$. Say $d(p_1,q_1) = d(p_1,q_2) = \cdots = d(p_1,q_r) = a$, for $Q = \{q_1, \dots, q_r\}$ a subset of $\{p_2, \dots p_n\}$. Then all points in Q lie in a semicircle of radius a centered at p_1 (as p_1 belongs to the convex hull). We may assume that $q_1, q_2, \dots, q_r$ are in this order on the semicircle, so we obtain $r-1$ distinct distances $d(q_1,q_2), d(q_1,q_3), \dots, d(q_1,q_r)$. Hence

$$f(N) \geq \max\{r-1, k\} \geq \max\left\{r-1, \frac{N-1}{r}\right\} \ ,$$

which is minimised when $r(r-1) = N-1$, so we obtain the desired result.

This is not an optimal approximation at all. Inspired by the example of the grid, Erdős provided the following conjecture:

Conjecture 1.2 (Erdős) *There exists a constant $C > 0$ such that $f(N) \geq C \cdot N^{1-\epsilon}$, for all $\epsilon > 0$.*

The lower bound for $f(N)$ has successively improved over the years. To give a few examples, Moser in [11] (1952) obtained that $f(N) \gtrsim N^{3/2}$, Chung, Szemerédi and Trotter in [1] (1992) obtained that $f(N) \gtrsim N^{4/5}/\log N$, and Kutz and Tardos in [10] (2004) obtained that $f(N) \gtrsim N^{0.864}$.

Erdős's conjecture remained open until 2010, when L. Guth and N.H. Katz in [7] proved the following theorem:

Theorem 1.3 *$f(N) \gtrsim N/\log N$.*

The main goal of this report is to give an exposition of the proof of theorem 1.3.

The proof is considered to be a very important breakthrough and it is very innovative. It mainly uses the polynomial method, a new and powerful technique in combinatorics. Other examples of famous conjectures recently proved using the polynomial method include the finite field Kakeya problem (Dvir [2] in 2009) or the joints problems (Guth and Katz [8] the 3-dimensional case and Kaplan, Sharir and Shustin [9] the n-dimensional case in 2010). What is remarkable about the polynomial method is that it provides short proofs to problems which were previously considered very hard and deep. Also, since it is a relatively new method, the full power of the polynomial method is not yet fully understood. It turned out that it has very many applications in additive combinatorics, incidence geometry and number theory.

There are some very good introductory resources to polynomial method. For example, Tao provided a very good expository article in [16], while I found particularly helpful the polynomial method course given by Guth at MIT in Fall 2012 (available in [5]). Also, last year Guth published a book about the polynomial method in combinatorics [6].

Next, we are going to discuss the main steps in the proof of theorem 1.3. We will more or less follow the approach from the original paper of Guth and Katz [7]. In section 2, we are going to show how to reduce our problem about distances between points in $\mathbb{R}^2$ to a problem about incidences of lines in $\mathbb{R}^3$, by a method which is now commonly called the Elekes-Sharir framework. We will show that in order to prove theorem 1.3, it will be enough to prove the following theorem:

Theorem 1.4 *Let $\mathscr{L}$ be a set of N^2 lines in $\mathbb{R}^3$ such that there are $\lesssim N$ lines in any plane or any degree 2 surface. Then for all $2 \leq k \leq N$, the number of points that belong to at least k lines is $\lesssim N^3k^{-2}$.*

The key idea is to find a polynomial $P \in \mathbb{R}[x, y, z]$ such that $Z(P)$ contains "most" of the lines in $\mathscr{L}$ (throughout this essay, by $Z(P)$ we mean the zero set of P, i.e. $\{a \in \mathbb{R}^3 : P(a) = 0\}$). We will see that we have to treat the cases $k = 2$ and $k \geq 3$ separately. This is because if there is a point $a \in Z(P)$ such that there are 3 lines in $Z(P)$ that intersect at a, then we can find some additional properties of a and apply them. Such points are called *flat* points and their theory is exposed in detail in section 3.2.

When $k = 2$, we will use the theory of ruled surfaces in 3 dimensions (a surface Z is ruled if through any point $a \in Z$, there is a line $l \subset Z$ such that $a \in l$). In section 3.3 we will prove 2 important properties of ruled surfaces. First we will see that if for some polynomial P, the surface $Z(P)$ contains many lines, then it must have a ruled factor. Secondly, we will prove that if $Z(P)$ is a ruled surface that has no factors which are planes or reguli, then the number of intersections between N^2 of lines in $Z(P)$ is $O(N^3)$. Reguli are doubly-ruled irreducible surfaces of degree 2 which we will describe properly in section 3.3. This theorem is important because we can see that in a plane or in a regulus, N^2 lines can have $\sim N^4$ points of intersections, so it means that planes and reguli are essentially the only surfaces with many points of intersections between lines.

We complete the proof for the case $k = 2$ in section 5. Using a probabilistic argument, we will find a polynomial P of low degree that contains most lines in $\mathscr{L}$ and most points of intersections between lines in $\mathscr{L}$. By assumption, there are not many lines in a plane or a regulus, so there are not many intersections inside planes or reguli. Also, the other components of P don't contribute with many intersections, so we'll be able to bound the number of intersections between lines in $\mathscr{L}$. Of course, everything will be treated and explained carefully.

For the case $k \geq 3$, we will first study the polynomial cell decomposition, which was the most celebrated idea in the proof of Guth and Katz in [7]. In section 4 we will show that given a number of points in $\mathbb{R}^3$, we can find a non-zero polynomial P of low degree and a cell decomposition of $\mathbb{R}^3$ such that the boundary of the the cells are contained in $Z(P)$ and that each cell contains "few" points. Also, we are going to show how to use this in order give a proof to the classical Szémeredi-Trotter theorem, probably the most important theorem and tool in incidence geometry.

We complete the proof in section 6. Using the cell decomposition method, we find a polynomial P such that $Z(P)$ contains most of the lines in $\mathscr{L}$ and most of points that belong to at least k lines of $\mathscr{L}$. Let's call S the set of points that belong to at least k lines of $\mathscr{L}$ and $S' \subset S$ the subset of these points which are contained in 3 lines of $\mathscr{L}$ belonging to $Z(P)$ (so points in S' are either critical or flat). We will show that S' contains most of the points of S and that most of the lines contain "many" points of S'. In section 3.2 we will see that if $Z(P)$ has no planar component, there cannot be many such lines, hence most lines will lie in planar components of $Z(P)$. We will obtain the conclusion from the fact that the we control the degree of P to be fairly low and that by assumptions there are not too many lines in a plane.

Note that theorem 1.3 does not fully solve the Erdős distinct distances problem, even if it makes substantial progress. All we know so far is that

$$\frac{N}{\log N} \lesssim f(N) \lesssim \frac{N}{\sqrt{\log N}} \ .$$

It is not currently known if there is a better example than the grid that provides a better upper bound for $f(N)$ or if the lower bound we prove in this report can be improved. We note that is unlikely that this lower bound can be improved using the polynomial method, because actually this approach is all about proving theorem 1.4 in incidence geometry which in turn using the Elekes-Sharir framework implies that $f(N) \gtrsim N/\log N$.

We now discuss some related open problems. One natural question to ask is the distinct distances problem in higher dimensions. Denote by $f_d(N)$ the minimal possible number of distinct distances among N points in $\mathbb{R}^d$. In the same paper from 1946 [4], Erdős showed the bounds

$$N^{1/d} \lesssim f_d(N) \lesssim N^{2/d} \ .$$

He conjectured that for $d \geq 3$, $f_d(N)$ behaves asymptotically as $N^{2/d}$.

Conjecture 1.5 (Erdős) *Let $d \geq 3$. Then there exists constants c_1 and c_2 such that*

$$c_1 N^{2/d} \leq f_d(N) \leq c_2 N^{2/d} \ .$$

This conjecture is still open. The best lower bound currently known was found by Solymosi and Vu in 2008 in [13], where they greatly improved the lower bound to

$$f_d(N) \gtrsim N^{\frac{2}{d} - \frac{2}{d(d+2)}} \ .$$

We note that the higher dimensional distinct distances problem is more difficult to attempt using the polynomial method, because the we cannot apply the Elekes-Sharir framework in order to transform it into an incidence geometry problem.

If we think about the minimum number of distances between N points in plane, it is also natural to think what is the maximum number of times a distance appears between N points in plane. This is known as the unit distances problem, and was posed in the same paper [4]. Denote by $u(N)$ the maximum number of pairs at unit distance in a set of N points in the plane. Erdős proved that there exists a constant c such that

$$N^{1+c/\log\log N} < u(N) < N^{3/2} \ .$$

He also conjectured that $u(N) = O(N^{1+\epsilon})$, for all $\epsilon > 0$. The best current upper bound is due to Spencer, Szemerédi and Trotter in [14] in 1984, where they proved that $u(N) = O(N^{4/3})$.

The main goal of this report is to provide an accessible and easy to read proof of theorem 1.3. In order do this, I chose to have all algebraic geometry prerequisites separately in section 3 where

I attempt to explain them in great detail. Also, I have chosen to work with explicit constants rather than O or $\lesssim$ notation most of the time just to make thinks clearer (this is because one of the things I found difficult in the original paper [7] was the independence of some inequalities involving $\lesssim$). Even though I worked with sharper bounds than in [7], I still obtained constants as big as 2^{70}, but they are nonetheless finite constants (and by no means the best obtainable, I was also rather careless in many of my approximations and inequalities).

2 Reduction to an incidence problem

The purpose of this section to connect our problem about the number of distinct distances to a problem of incidences of lines in $\mathbb{R}^3$.

Let $\mathscr{P} \subset \mathbb{R}^2$ denote a set of N points in the plane. We would like to look at the set of distances between them

$$d(\mathscr{P}) := \{d(p,q) : p, q \in \mathscr{P}, p \neq q\} \ .$$

Our main goal is to show that $d(P) \gtrsim N/\log N$. We will look at the set of quadruples

$$Q(\mathscr{P}) = \{(p,q,r,s) \in \mathscr{P}^4 : d(p,q) = d(r,s) \neq 0\} \ .$$

Intuitively, if the set of distinct distances $d(\mathscr{P})$ is small, then we would have many distances repeated, so $Q(\mathscr{P})$ should be large. Indeed, we easily obtain the following inequality:

Lemma 2.1 *Let $P \subset \mathbb{R}^2$ a set of N points in the plane. Then*

$$|d(\mathscr{P})| \geq (N^2 - N)^2 / |Q(\mathscr{P})| \ .$$

Proof Say $d(\mathscr{P}) = \{d_1, \dots, d_m\}$, where $m = |d(\mathscr{P})|$. For each $1 \leq i \leq m$, say there are n_i pairs $(p,q) \in \mathscr{P}^2$ such that $d(p,q) = d_i$. The total number of pairs (p,q) such that $p \neq q$ is $N^2 - N$, so clearly we have

$$\sum_{i=1}^{m} n_i = N^2 - N \ .$$

On the other hand, we note that

$$|Q(\mathscr{P})| = \sum_{i=1}^{m} n_i^2 \ .$$

Hence, by applying Cauchy-Schwarz, we obtain

$$|Q(\mathscr{P})| = \sum_{i=1}^{m} n_i^2 \geq \frac{1}{m} \left(\sum_{i=1}^{m} n_i \right)^2 \geq \frac{(N^2 - N)^2}{|d(\mathscr{P})|} \ .$$

Rearranging, we obtain the desired conclusion. ∎

Now we note that in order to prove that $d(\mathscr{P}) \gtrsim N/\log N$, it is enough to prove that $|Q(P)| = O(N^3 \log N)$. Indeed, suppose there exists an universal constant C such that $|Q(\mathscr{P})| \leq CN^3 \log N$, for all sets $\mathscr{P} \subset \mathbb{R}^2$ of N points in plane. Then we have

$$|d(\mathscr{P})| \geq \frac{(N^2 - N)^2}{C \cdot N^3 \log N} \geq \frac{1}{4C} N/\log N \ ,$$

because $(N^2 - N)^2 \geq (1/4)N^4$ for all $N \geq 2$.

Now we focus on describing $Q(\mathscr{P})$. For this purpose, let G denote the group of orientation-preserving rigid motions of the plane, that is the group generated by rotations and translations. This is useful because of the following observation:

Lemma 2.2 *Let p, q, r and s be 4 points in $\mathbb{R}^2$ such that $p \neq q$. Then $d(p,q) = d(r,s)$ if and only if there exists a unique $g \in G$ such that $g(p) = r$ and $g(q) = s$.*

Proof Let p, q, r and s be 4 points in $\mathbb{R}^2$ such that $d(p,q) = d(r,s) \neq 0$. Note that all rigid motions in G sending p to r are of the form $R_{r,\theta} \circ \tau_{r-p}$, where $R_{r,\theta}$ is the rotation of angle θ about the point r, and τ_{r-p} is the translation sending p to r (indeed, check that if g is a positive isometry such that $g(p) = r$, then $g \circ \tau_{r-p}^{-1}(r) = r$, so $g \circ \tau_{r-p}^{-1}$ must be a rotation around r). Also, since $g(q) = s$, we have that

$$s = g(q) = R_{r,\theta} \circ \tau_{r-p}(q) = R_{r,\theta}(q + r - p) \ .$$

Since $d(r, q+r-p) = |q-p| = |r-s| = d(r,s)$, there exists a unique rotation around r sending $q + r - p$ to s.

The other direction is trivial, since g is a rigid motion, so it preserves distances. ∎

The lemma above shows that for each quadruple $(p, q, r, s) \in Q(\mathscr{P})$, there exists a unique $g \in G$ such that $g(p) = r$ and $g(q) = s$. Hence we can define a map

$$E : Q(\mathscr{P}) \to G$$

which sends every quadruple to the corresponding rigid motion. Then clearly we have

$$|Q(\mathscr{P})| = \sum_{g \in G} |E^{-1}(g)| \ ,$$

where $E^{-1}(g)$ is non-empty for finitely many positively oriented rigid motions. Therefore it would be useful to find an estimate for $|E^{-1}(g)|$. It is not hard to observe that $|E^{-1}(g)|$ depends on the number of points in $\mathscr{P}$ preserved by g (which is $|\mathscr{P} \cap g\mathscr{P}|$).

Lemma 2.3 *Say $g \in G$ and $|\mathscr{P} \cap g\mathscr{P}| = k$. If $k < 2$, then $|E^{-1}(g)| = 0$ and if $k \geq 2$, then $|E^{-1}(g)| = k(k-1)$.*

Proof Let $(p, q, r, s) \in Q(\mathscr{P})$ and $g = E((p, q, r, s))$. Then $g(p) = r$ and $g(q) = s$, hence $\{r, s\} \subset \mathscr{P} \cap g\mathscr{P}$. Therefore $|\mathscr{P} \cap g\mathscr{P}| \geq 2$ as $r \neq s$ by definition.

Now suppose that $|\mathscr{P} \cap g\mathscr{P}| = \{r_1, \ldots, r_k\}$, for some $k \geq 2$. Let $p_i = g^{-1}(r_i)$, for $1 \leq i \leq k$. Then clearly $p_1, \ldots, p_k \in P$ and they are distinct. It is easy to see that $(p_i, p_j, r_i, r_j) \in E^{-1}(g)$, for all $1 \leq i, j \leq k$ with $i \neq j$, as g preserves distances. This gives rise to $k(k-1)$ quadruples in $E^{-1}(g)$. We need to check these are all of them.

Let $(p, q, r, s) \in Q(\mathscr{P})$ such that $g = E((p, q, r, s))$. Then clearly, as we noted before,

$$\{r, s\} \subset \mathscr{P} \cap g\mathscr{P} \ .$$

Say $r = r_i$ and $s = r_j$, for some i, j. Then $p = g^{-1}(r) = p_i$ and $q = g^{-1}(s) = p_j$, so it is one of the quadruples described above. ∎

So it is natural to define

$$G_{=k} := \{g \in G : |\mathscr{P} \cap g\mathscr{P}| = k\}$$

and

$$G_k := \{g \in G : |\mathscr{P} \cap g\mathscr{P}| \geq k\} \ .$$

Using the lemma above, we obtain

$$
\begin{aligned}
|Q(\mathscr{P})| = \sum_{g\in G} |E^{-1}(g)| &= \sum_{k=2}^{N} |G_{=k}| \cdot k(k-1) \\
&= \sum_{k=2}^{N} (|G_k| - |G_{k+1}|) \cdot k(k-1) \quad (\text{as } G_{N+1} = \emptyset) \\
&= \sum_{k=2}^{N} |G_k| \cdot ((k(k-1) - (k-1)(k-2)) \\
&= \sum_{k=2}^{N} |G_k| \cdot (2k-2)
\end{aligned}
$$

Hence we want to bound $|G_k|$. This suggest the following theorem:

Theorem 2.4 *There exists an universal constant $C > 0$ such that for all N and for any set $\mathscr{P} \subset \mathbb{R}^2$ of N points in plane, we have*

$$|G_k| \le C \cdot N^3 k^{-2} \ ,$$

for all $2 \le k \le N$.

It is easy to see that this would imply that $Q(\mathscr{P}) = O(N^3 \log N)$, because

$$|Q(\mathscr{P})| \le \sum_{k=2}^{N} |G_k| \cdot (2k-2) \le 2CN^3 \cdot \sum_{k=2}^{N} \frac{1}{k} \le 4C \cdot N^3 \log N \ .$$

Hence now the main we focus on proving theorem 2.4. We will first prove it for translations. Let $T \subset G$ denote the subgroup of G of translations, which is isomorphic to $\mathbb{R}^2$. We obtain:

Lemma 2.5 *For any set $\mathscr{P} \subset \mathbb{R}^2$ of N points and for all $2 \le k \le N$, we have*

$$|G_k \cap T| \le 2N^3 k^{-2} \ .$$

Proof First, we notice that if g is a translation, then $g(x) = x + v$, for some $v \in \mathbb{R}^2$ and if $(p, q, r, s) \in E^{-1}(g)$, then $p + v = r$ and $q + v = s$, hence $r - p = s - q$. So if we define

$$Q_T(\mathscr{P}) := E^{-1}(T) = \{(p, q, r, s) \in Q(\mathscr{P}) : r - p = s - q\} \ ,$$

we note that $|Q_T(\mathscr{P})| \le N^3$, because if $(p, q, r, s) \in Q_T(\mathscr{P})$, s is uniquely determined by p, q and r (in fact $s = r - p + q$).

Using lemma 2.3, and proceeding similar as before, we note that

$$
\begin{aligned}
N^3 \ge |Q_T(\mathscr{P})| = \sum_{g\in T} |E^{-1}(g)| &= \sum_{j=2}^{N} |G_{=j} \cap T| \cdot j(j-1) \\
&\ge \sum_{j=k}^{N} |G_{=j} \cap T| \cdot j(j-1) \\
&\ge k(k-1) \cdot \sum_{j=k}^{N} |G_{=j} \cap T| = k(k-1)|G_k \cap T|
\end{aligned}
$$

Hence

$$|G_k \cap T| \le \frac{N^3}{k(k-1)} \le \frac{2N^3}{k^2} \ . \quad \blacksquare$$

Let $G' = G \setminus T$. Therefore we want to show $|G_k \cap G'| \lesssim N^3 k^{-2}$. We observe that G' can be viewed as a set of rotations about a unique fixed point $(x, y) \in \mathbb{R}^2$ by a unique angle $\theta \in (0, 2\pi)$ (the angle 0 is excluded because the identity rigid motion was accounted as a translation). Hence we have the bijection $\rho : G' \to \mathbb{R}^3$ defined by

$$\rho(g) = (x, y, \cot(\theta/2)) \ ,$$

where (x, y) is the unique fixed point of g and θ is the angle of rotation.

We want to justify this definition. Denote

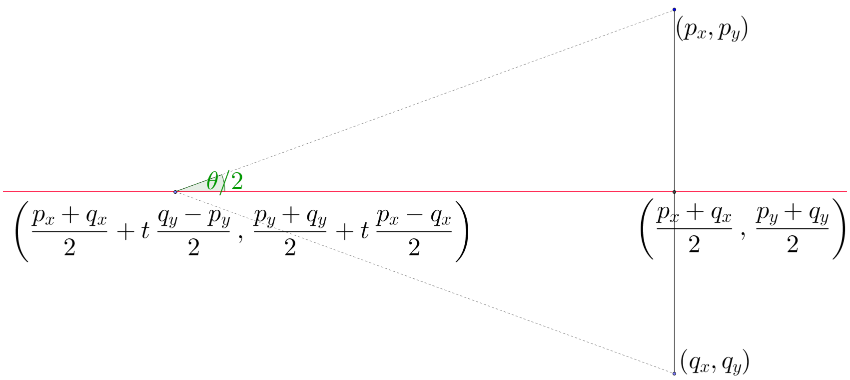
$$S_{pq} = \{g \in G : g(p) = q\}$$

which is the set of all positively-oriented rigid motions sending p to q. We will show that the set $S_{pq} \cap G'$ is sent by ρ to a line in $\mathbb{R}^3$. This will allow to transform our problem in a problem about incidences about lines in $\mathbb{R}^3$.

Lemma 2.6 *Let $p = (p_x, p_y)$ and $q = (q_x, q_y)$ be two points in $\mathbb{R}^2$ (not necessarily distinct). Then $\rho(S_{pq} \cap G')$ is a line in $\mathbb{R}^3$ which can be parameterised as*

$$l = \left\{ \left(\frac{p_x + q_x}{2}, \frac{p_y + q_y}{2}, 0 \right) + t \left(\frac{q_y - p_y}{2}, \frac{p_x - q_x}{2}, 1 \right) : t \in \mathbb{R} \right\} \tag{1}$$

Proof We note that the fixed point of any rotation sending p to q must lie on the perpendicular bisector of p and q.



Let r be a point on the perpendicular bisector of p and q. Then

$$r = \left(\frac{p_x + q_x}{2} + t\frac{q_y - p_y}{2}, \frac{p_y + q_y}{2} + t\frac{p_x - q_x}{2} \right)$$

for some $t \in \mathbb{R}$. Then there is a unique rotation centered at r which sends p to q. Call it g_r. Say θ is the angle of this rotation. Then, by definition of $\cot(\theta/2)$, we obtain

$$\cot(\theta/2) = \frac{\left|r - \frac{p+q}{2}\right|}{\left|\frac{p-q}{2}\right|} = \frac{t\left(\left(\frac{q_y - p_y}{2}\right)^2 + \left(\frac{p_x - q_x}{2}\right)^2\right)^{1/2}}{\left(\left(\frac{p_x - q_x}{2}\right)^2 + \left(\frac{p_y - q_y}{2}\right)^2\right)^{1/2}} = t \ .$$

So indeed we obtain that

$$\rho(g_r) = \left(\frac{p_x + q_x}{2} + t\frac{q_y - p_y}{2}, \frac{p_y + q_y}{2} + t\frac{p_x - q_x}{2}, t \right) \ ,$$

which gives us the desired result. ∎

For any two points $p, q \in \mathbb{R}^2$, denote

$$L_{pq} = \rho(S_{pq} \cap \mathbb{R}^2) \ .$$

We now state and prove some easy facts about these lines.

Lemma 2.7 *For any 4 points p, q, r and s in $\mathbb{R}^2$, the lines L_{pq} and L_{rs} coincide if and only if $p = r$ and $s = q$.*

Proof Using the parameterisation given by (1), we note that if $L_{pq} = L_{rs} = l$, first by looking at the intersection of l with the plane $z = 0$, we obtain $\left(\frac{p_x+q_x}{2}, \frac{p_y+q_y}{2}, 0\right) = \left(\frac{r_x+s_x}{2}, \frac{r_y+s_y}{2}, 0\right)$. It follows that $\left(\frac{q_y-p_y}{2}, \frac{p_x-q_x}{2}, 1\right) = \left(\frac{s_y-r_y}{2}, \frac{r_x-s_x}{2}, 1\right)$, hence $p = r$ and $s = q$. ∎

Lemma 2.8 *If $p, q, r \in \mathbb{R}^2$ such that $q \neq r$, then the lines L_{pq} and L_{pr} are skew.*

Proof Suppose we have a point $a \in L_{pq} \cap L_{pr}$. Say $a = (a_1, a_2, a_3)$. Then, by looking at (1), we have that

$$\begin{aligned} a_1 &= \frac{p_x + q_x}{2} + a_3\frac{q_y - p_y}{2} = \frac{p_x + r_x}{2} + a_3\frac{r_y - p_y}{2} \\ a_2 &= \frac{p_y + q_y}{2} + a_3\frac{p_x - q_x}{2} = \frac{p_y + r_y}{2} + a_3\frac{p_x - r_x}{2} \end{aligned}$$

This implies that

$$\begin{aligned} q_x + a_3 q_y &= r_x + a_3 r_y \\ q_y - a_3 q_x &= r_y - a_3 r_x \end{aligned}$$

Hence $q_x = r_x$ and $q_y = r_y$, hence $q = r$.

Also, if L_{pq} and L_{pr} are parallel, then they must have the same direction, so again the parameterisation (1) implies that

$$\left(\frac{q_y - p_y}{2}, \frac{p_x - q_x}{2}, 1\right) = \left(\frac{r_y - p_y}{2}, \frac{p_x - r_x}{2}, 1\right) \ ,$$

hence $q = r$. ∎

Let $a \in \mathbb{R}^2$ be a point. We denote

$$\mathscr{L}_a := \{L_{ap} : p \in \mathbb{R}^2\} \ .$$

We've just seen that $\mathscr{L}_a$ is a set of pairwise-skew lines.

Lemma 2.9 *Fix a point $a \in \mathbb{R}^2$. Then every point in $\mathbb{R}^3$ belongs to exactly one line in $\mathscr{L}_a$. Moreover, we can construct a vector field*

$$V = (V_1(x, y, z), V_2(x, y, z), V_3(x, y, z))$$

on $\mathbb{R}_3$ which is tangent at every point to the unique line in $\mathscr{L}_a$ and each of the coefficients V_1, V_2 and V_3 are polynomials of degree at most 2.

Proof Let (x, y, z) be a point in $\mathbb{R}^3$. We want to check it lies in exactly one line of $\mathscr{L}_a$. Using (1), (x, y, z) belongs to the line L_{ap} if and only if for some $t \in \mathbb{R}$, we have that

$$\left(\frac{a_x + p_x}{2}, \frac{a_y + p_y}{2}, 0\right) + t\left(\frac{p_y - a_y}{2}, \frac{a_x - p_x}{2}, 1\right) = (x, y, z) \ .$$

We note that we must have $t = z$, and the following equations must hold:

$$p_x + zp_y = 2x - a_x + a_y z$$
$$-zp_x + p_y = 2y - a_y - a_x z$$

We treat a_x and a_y as constants. Hence we obtain

$$p_x = \frac{2x - a_x + a_y z - 2yz + a_y z + a_x z^2}{1 + z^2}$$
$$p_y = \frac{2y - a_y - a_x z + 2xz - a_x z + a_y z^2}{1 + z^2}$$

So indeed we can solve the equation uniquely for p_x and p_y, hence (x, y, z) belongs on a unique line L_{ap}.

Now we define the the vector field

$$\begin{aligned} V(x, y, z) &= (1 + z^2)\left(\frac{p_y - a_y}{2}, \frac{a_x - p_x}{2}, 1\right) \\ &= \left(y - a_y - a_x z + xz, -x + a_x - a_y z + yz, 1 + z^2\right) \end{aligned}$$

which is tangent to the direction of L_{ap} and each component is a polynomial of degree 2. ∎

We now put together all the information we have acquired so far. Recall that our goal was to prove that $|G_k \cap G'| \lesssim N^3 k^{-2}$. Let $\mathscr{L} = \{L_{pq} : p, q \in \mathscr{P}\}$, which by lemma 2.7 is a set of N^2 distinct lines in $\mathbb{R}^3$.

Let $g \in G_k \cap G'$. Then there exists a set of k distinct points $\{q_1, \ldots, q_k\} \subset \mathscr{P} \cap g\mathscr{P}$. Denote

$$p_i = g^{-1}(q_i)\ ,$$

for all $1 \leq i \leq k$. As g is a rigid motion, then all p_i's are distinct. By definition,

$$g \in S_{p_i g_i} \cap G'\ ,\ \text{for all } i.$$

Hence $\rho(g)$ belongs to all lines $L_{p_i g_i}$. So if $g \in G_k \cap G'$, then $\rho(g)$ belongs to at least k lines in $\mathscr{L}$. This means it would be enough to show there are $\lesssim N^3 k^{-2}$ points which belong to at least k lines in $\mathscr{L}$. Also note that lemma 2.8 implies that a plane contains at most N lines of $\mathscr{L}$. Indeed, suppose there exists a plane that contains more than N lines of $\mathscr{L}$. Then we can find $p, q, r \in P$ with $q \neq r$ such that L_{pq} and L_{pr} belong to the same plane, which is a contradiction to lemma 2.8.

Indeed, we will prove the following theorem:

Theorem 2.10 *Let $\mathscr{L}$ be a set of N^2 lines in $\mathbb{R}^3$ such that any plane contains at most N of the lines in $\mathscr{L}$ and let $3 \leq k \leq N$. Then there exists an universal constant C such that the number of points that belong to at least k lines in $\mathscr{L}$ is at most CN^3k^{-2}.*

The proof of this theorem will be the objective of section 5. Note that in the statement of the theorem we have assumed that $k \geq 3$. This is because the statement for $k = 2$ is not necessarily true. We will see later that it is possible to have N^2 lines in $\mathbb{R}^3$ with no N in any plane such that the number of intersections between them is $\sim N^4$. Hence in order to obtain a result about the case $k = 2$, we will have to make one extra assumption about the set $\mathscr{L}$. We will later show that it also holds that $\mathscr{L}$ contains $\lesssim N$ lines in any regulus. A regulus is a doubly-ruled surface of degree 2. We will properly define reguli, discuss some properties of them later in the paper. We will prove the following theorem:

Theorem 2.11 *Let $\mathscr{L}$ be a set of N^2 lines in $\mathbb{R}^3$ such that any plane contains at most N lines of $\mathscr{L}$ and any regulus at most $O(N)$ of lines in $\mathscr{L}$. Then the number of points of intersection between the lines in $\mathscr{L}$ is $\lesssim N^3$.*

Hence, if we assume theorems 2.10 and 2.11 hold, we obtain that $|G_k \cap G'| \lesssim N^3 k^{-2}$, which in turn implies $Q(\mathscr{P}) \lesssim N^3 \log N$, as we've seen earlier. In the rest of the paper we will develop methods to prove these 2 theorems.

3 Algebraic geometry preliminaries

3.1 Zero set of polynomials

In this subsection we will look at some classical results about polynomials we will repeatedly use in the sections to follow. In some sense, these kind of results are the basis of the theory we are going to build on.

We begin by recalling some 3 veriations of the classical Bézout theorem.

Lemma 3.1 (Bézout) *Let $P \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial of degree d and let l be a line in $\mathbb{R}^n$. Then if $|Z(P) \cap l| > d$, then $l \subset Z(P)$.*

Proof Say $l = \{a + t\mathbf{v} : t \in \mathbb{R}\}$, where $\mathbf{v}$ if the direction of l. Denote

$$Q(t) = P(a + t\mathbf{v}) \ .$$

Then Q is a one dimensional polynomial of degree d. If $|Z(P) \cap l| > d$, it means that Q vanishes at more than d values of t. Hence $Q \equiv 0$, which means that P vanishes on l. ∎

Remark Informally, the lemma above means that if a polynomial of degree vanishes in more than d points of a line, then the line must be included in the algebraic surface $Z(P)$.

Theorem 3.2 (Bézout) *Let $P, Q \in \mathbb{R}[x, y]$ be non-zero polynomials of degrees m and n respectively. If there are more than mn points in $\mathbb{R}^2$ where both P and Q vanish, then P and Q have a common factor.*

Also, we would like to highlight the 3-dimensional version of this theorem:

Theorem 3.3 (Bézout) *Let $P, Q \in \mathbb{R}[x, y, z]$ be non-zero polynomials of degrees m and n respectively. If there are at least $mn + 1$ different lines in $\mathbb{R}^3$ on which P and Q simultaneously vanish, then P and Q have a common factor.*

We will not provide proofs for the previous two theorems, as they are classical results which can be found in any first course in algebraic geometry. The proofs are rather tedious and outside the scope of this report.

Next, we would like to show that if we are given N points in $\mathbb{R}^n$, we can find a non-zero polynomial of "low" degree such that all N points are contained in the zero set of the polynomial. We begin with the following crude bound:

Lemma 3.4 *Let S be a set of N points in $\mathbb{R}^n$. Then if $N < \binom{n+d}{d}$, for some integer d, then there exists a non-zero polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most d that vanishes on all the points of S.*

Proof Let V be the vector space of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ of degree at most d, which has dimension $m = \binom{n+d}{d}$ over $\mathbb{R}$. A basis for V is formed by monomials of the form

$$x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \quad \text{where } \alpha_1 + \ldots \alpha_n \leq d \ .$$

Call this monomials $P_1, \ldots P_m$. Say $S = \{a_1, \ldots, a_N\}$ is our set of N points in $\mathbb{R}^n$, with $N < m$. For $1 \leq i \leq m$, define $\mathbf{v}_i \in \mathbb{R}^N$ by

$$\mathbf{v}_i = (P_i(a_1), P_i(a_2), \ldots, P_i(a_N)) \ .$$

So we have a set of m vectors in $\mathbb{R}^N$ and $m > N$, so these vectors are not linearly independent, so there exists $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$, not all 0, such that

$$\sum_{i=1}^{m} \gamma_i \mathbf{v}_i = \mathbf{0} \ .$$

Denote $P = \gamma_1 P_1 + \cdots + \gamma_m P_m$. Then clearly $P(a_j) = 0$, for all $1 \leq j \leq N$ and P has degree at most d. ∎

From now on, we focus on trivariate polynomials $P \in \mathbb{R}[x, y, z]$. We would like to obtain a similar result as in the previous lemma, but for lines.

Lemma 3.5 *Let $\mathscr{L}$ be a set of N lines in $\mathbb{R}^3$. Then there exists a non-zero trivariate polynomial of degree at most $4N^{1/2}$ that vanishes on each line.*

Proof Let d be a number we will choose later. Choose $d+1$ points on each line in $\mathscr{L}$. Hence using lemma 3.4, if $\binom{d+3}{3} > (d+1)N$, then we find a non-zero polynomial of degree $\leq d$ that vanishes on on $d+1$ points of each line in L, so by using lemma 3.1, it would vanish on all the lines in $\mathscr{L}$. Since $\binom{d+3}{3} > \frac{d^3}{6}$, it is enough to check that $\frac{d^3}{6} > 2dN$, which is clearly true for $d = 4N^{1/2}$. ∎

Next we study the conditions under which a line is contained in the zero set of a polynomial.

Let P be a trivariate polynomial of degree d. We have the Taylor expansion

$$P(x, y, z) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k \leq d}} \frac{1}{i!j!k!}(x_1 - a_1)^i (y - a_2)^j (z - a_3)^k \frac{\partial^{i+j+k} P}{\partial x^i \partial y^j \partial z^k}(a_1, a_2, a_3) \tag{2}$$

It would be useful to define the t-th order of this expansion, for all $1 \leq t \leq d$. Therefore we define the following polynomials in 6 variables:

$$P_t(\mathbf{x}, \mathbf{v}) = P_t(x_1, x_2, x_3, v_1, v_2, v_3) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k = t}} \frac{1}{i!j!k!} v_1^i v_2^j v_3^k \frac{\partial^{i+j+k} P}{\partial x^i \partial y^j \partial z^k}(x, y, z) \tag{3}$$

for all $1 \leq t \leq d$. For example,

$$P_1(a, v) = \nabla P(a) \cdot v$$

and

$$P_2(a, v) = \frac{1}{2} v^T H_P(a) v \ ,$$

where $H_P(a)$ is the Hessian matrix evaluated at a.

$$H_P(a) = \begin{pmatrix} P_{xx}(a) & P_{xy}(a) & P_{xz}(a) \\ P_{xy}(a) & P_{yy}(a) & P_{yz}(a) \\ P_{xz}(a) & P_{yz}(a) & P_{zz}(a) \end{pmatrix}$$

Therefore we can rewrite the Taylor expansion as

$$P(a + v) = P(a) + \sum_{t=1}^{d} P_t(a, v) \tag{4}$$

Lemma 3.6 *Let P be a polynomial of degree d. Then a line l is contained in $Z(P)$ if and only if it is contained to order d at one if its points (i.e. exists $a \in l$ such that*

$$P(a) = P_1(a, \mathbf{v}) = \cdots = P_d(a, \mathbf{v}) = 0 \ ,$$

where $\mathbf{v}$ is the direction of l).

Proof Suppose $l \subset Z(P)$. Pick any $a \in l$ and let $\epsilon > 0$ small enough. It follows that

$$0 = P(a + \epsilon \mathbf{v}) = \epsilon P_1(a, \mathbf{v}) + o(\epsilon)$$

which implies $P_1(a, \mathbf{v}) = 0$. Proceed by induction to obtain

$$0 = P(a + \epsilon \mathbf{v}) = \epsilon^t P_t(a, \mathbf{v}) + o(\epsilon^t)$$

for all $1 \le t \le d$, so the claim follows.

Now suppose there exists $a \in l$ such that $P(a) = P_1(a, \mathbf{v}) = \cdots = P_d(a, \mathbf{v}) = 0$. Clearly, for all $t \in \mathbb{R}$, we have

$$P(a + t\mathbf{v}) = P(a) + \sum_{j=1}^{d} t^j P_j(a, \mathbf{v}) = 0 \ . \qquad \blacksquare$$

3.2 Critical points and flat points

Definition Given a polynomial $P \in \mathbb{R}[x_1, \dots, x_n]$, a point $a \in Z(P)$ is called *critical* if

$$P(a) = \frac{\partial P}{\partial x_1}(a) = \frac{\partial P}{\partial x_2} = \cdots = \frac{\partial P}{\partial x_n} = 0 \ .$$

Hence a point a is critical if $P(a) = 0$ and $\nabla P(a) = \mathbf{0}$. If $a \in Z(P)$ is not critical, it is called *regular.* A line is *critical* if all its points are critical.

Lemma 3.7 *Let P a polynomial of degree d. Suppose that a line l contains more than d critical points of P. Then l is a critical line.*

Proof At each critical point a, $P(a) = P_{x_1}(a) = \cdots = P_{x_n}(a) = 0$. The proof follows easily from lemma 3.1. Note that since $|Z(P) \cap l| > d$ and $\deg(P) = d$, we must have $l \subset Z(P)$. Similarly, by noticing that $\deg(P_{x_i}) = d - 1$, for all $1 \le i \le n$, we get that $l \subset Z(P_{x_i})$. So P and ∇P vanish simultaneously on l, it follows that l is a critical line for P.

Lemma 3.8 *An irreducible trivariate polynomial P of degree d can have at most $d(d-1)$ critical lines.*

Proof This fact follows easily from Bézout's theorem 3.3. We note that the degree of each component of ∇P is $d - 1$, so if $Z(P)$ and $Z(P_{x_i})$ have more than $d(d-1)$ common lines, then they must have a common factor, which contradicts irreducibility of P. $\blacksquare$

We can generalise this fact to square-free polynomials.

Lemma 3.9 *An square-free trivariate polynomial P of degree d can have at most $d(d-1)$ critical lines.*

Proof We will proceed by induction on d. If P is irreducible, we are done by the previous lemma. So suppose $P = P_1 P_2$, where P_1 and P_2 are square-free polynomials with no common factors of degrees d_1, respectively d_2. Clearly $d_1, d_2 \ge 1$ and $d_1 + d_2 = d$.

Let l be a critical line of P. Using Lemma 3.1, we observe that l must be contained in at least one of $Z(P_1)$ or $Z(P_2)$ (because $P_1 P_2$ vanish on l). Also, we note that

$$\nabla P = P_1 \nabla P_2 + \nabla P_1 P_2 \ .$$

So we must be in at least one of the following cases:

1. $l \in Z(P_1) \cap Z(P_2)$

2. l is a critical line for P_1
3. l is a critical line for P_2

Indeed, suppose without losing the generality that $l \in Z(P_1)$, but $l \notin Z(P_2)$. Then $|l \cap Z(P_2)| \leq d_2$. So ∇P_1 vanishes on infinity many points of l, hence $l \subset Z(\nabla P_1)$.

We observe that there are at most $d_1 d_2$ lines in $Z(P_1) \cap Z(P_2)$, as they have no common factor (again using 3.3). By induction hypothesis, P_1 has at most $d_1(d_1 - 1)$ critical lines, and similar estimate holds for P_2. Therefore, the total number of critical lines is at most

$$d_1 d_2 + d_1(d_1 - 1) + d_2(d_2 - 1) \leq (d_1 + d_2)(d_1 + d_2 - 1) = d(d - 1) \quad \blacksquare$$

We now turn our attention to 3 dimensions.

Definition If $a \in Z(P)$ is a regular point, define π_a to be the tangent plane to $Z(P)$ at a. Hence

$$x \in \pi_a \iff (x - a) \cdot \nabla P(a) = 0 \ .$$

Lemma 3.10 *Let a be a regular point of P such that there exists a line l passing through a such that P vanishes on l. Then l is contained in π_a.*

Proof Clearly the directional derivative along l at a is 0. Say

$$l = \{a + t\mathbf{v} : t \in \mathbb{R}\}$$

for some direction $\mathbf{v}$. Then $\nabla_{\mathbf{v}}(a) = 0$. But $\nabla_{\mathbf{v}}(a) = \nabla P(a) \cdot \mathbf{v}$. Hence l is contained in the tangent plane of P at a. $\blacksquare$

We now look at the second-order approximation of $P \in \mathbb{R}[x, y, z]$ at a given by Taylor expansion

$$Q_a(u) = P(a) + \nabla P(a) \cdot (u - a) + \frac{1}{2}(u - a)^T H_P(a)(u - a) \tag{5}$$

where $H_P(a)$ is the Hessian matrix

$$H_P(a) = \begin{pmatrix} P_{xx}(a) & P_{xy}(a) & P_{xz}(a) \\ P_{xy}(a) & P_{yy}(a) & P_{yz}(a) \\ P_{xz}(a) & P_{yz}(a) & P_{zz}(a) \end{pmatrix}$$

Clearly, if $a \in Z(P)$ is a regular point, then for $u \in \pi_a$ we have

$$Q_a(u) = \frac{1}{2}(u - a)^T H_P(a)(u - a) \ .$$

Definition Let $a \in Z(P)$ a regular point. We say a is a *flat point* of P if Q_a vanishes on π_a. We call a line l in $\mathbb{R}^3$ *flat* for a trivariate polynomial P if all the points on l are flat points for P (with the possible exception of finitely many critical points).

Remark We can regard the definition of a flat point as saying that the second degree approximation of a polynomial at one of its flat points is a plane.

Next we will derive some results about flat lines and points. The next lemma says that if a point belongs to at least three lines in $Z(P)$, then it is flat. Intuitively, this suggest why the study of flat points and lines will be helpful in the study of configurations of points and lines with many incidences.

Lemma 3.11 *Let P be a trivariate polynomial and let $a \in Z(P)$ be a regular point such that it is contained in three distinct lines on which P vanishes. Then a is a flat point for P.*

Proof Let l_1, l_2, l_3 be the three incident lines at a on which P vanishes, and denote $\mathbf{v}_1, \mathbf{v}_2$, respectively $\mathbf{v}_3$ their directions, where $|\mathbf{v}_i| = 1$, for $i = 1, 2, 3$. Lemma 3.10 implies that the lines l_1, l_2 and l_3 must belong to π_a.

First, we check that Q_a vanishes on the lines l_i, for $i = 1, 2, 3$. We know each $l_i \subset \pi_a$, hence

$$P(a) = \nabla P(a) \cdot \mathbf{v}_i = 0 \ .$$

Substitute $u = a + \epsilon \mathbf{v}_i$ (for ϵ sufficiently small) in the definition of Q_a to obtain

$$0 = P(a + \epsilon \mathbf{v}_i) = Q_a(a + \epsilon \mathbf{v}_i) + o(\epsilon^2) = \frac{\epsilon^2}{2} \mathbf{v}_i^T H_P(a) \mathbf{v}_i + o(\epsilon^2)$$

So we must have that $\mathbf{v}_i^T H_P(a) \mathbf{v}_i = 0$ for each i, so indeed Q vanishes on the lines l_i.

Now take any line $l \in \pi_a$ that does not passes through a and is not parallel to any of the l_i's. Because l_1, l_2 and l_3 are distinct and incident at a, then l intersects each l_i at three distinct points. But Q_a is a degree 2 polynomial and vanishes in at least 3 points of l, so by Bézout's theorem we must have $l \in Z(Q_a)$. This clearly implies $\pi_a \subset Z(Q_a)$. ∎

The argument above shows that that if Q_a vanishes at three points $u_i = a + t_i \mathbf{v}_i \in \pi_a$ ($t_i \neq 0$), where $\mathbf{v}_i$ are distinct, then $Q_a \equiv 0$ on π_a. This is because

$$Q_a(u_i) = 0 \Longleftrightarrow t_i^2 \mathbf{v}_i^T H_P(a) \mathbf{v}_i = 0 \iff \mathbf{v}_i^T H_P(a) \mathbf{v}_i = 0$$

so Q_a will vanish on the three lines $\{a + t\mathbf{v}_i : t \in \mathbb{R}\}$ and as in the proof of the lemma, this implies Q_a vanish on the whole π_a.

This suggests to look at the three points

$$u_i = a + \nabla P(a) \times e_i$$

which clearly belong to π_a. We have to assume we are in general position, so that π_a is not parallel to any of the 3 coordinate directions. So our 3 points are distinct and different from a.

To check that Q_a vanishes on the points u_i, it is enough to check that

$$(\nabla P(a) \times e_i)^T H_P(a) (\nabla P(a) \times e_i) = 0$$

for $i = 1, 2, 3$. Now we define the 3 polynomials

$$\Pi_i(P)(u) = (\nabla P(u) \times e_i)^T H_P(u) (\nabla P(u) \times e_i) \tag{6}$$

This are polynomials of degree at most $(d-1) + (d-2) + (d-1) = 3d - 4$.

Putting everything together, we deduce that if $\Pi_1(P)(a) = \Pi_2(P)(a) = \Pi_3(P)(a) = 0$, then Q_a will vanish on π_a, hence a is flat. We state this fact in the following lemma:

Lemma 3.12 *Let P be a trivariate polynomial and a be a regular point for P in general position. Then a is flat if and only if*

$$\Pi_1(P)(a) = \Pi_2(P)(a) = \Pi_3(P)(a) = 0 \ .$$

Proof We have already proved above that if the polynomials $\Pi_i(P)$ vanish at a regular point a in general position, then a is flat. For the other direction, assume a flat. Clearly

$$u_i = a + \nabla P(a) \times e_i \in \pi_a \ ,$$

hence $Q_a(u_i) = 0$, which is equivalent to $\Pi_i(P)(a) = 0$. It is interesting to note that for this direction we don't have to assume a is in general position. ∎

Lemma 3.13 *Let P be a trivariate polynomial of degree d and l a line in general position (not parallel to any of the 3 planes given by the coordinates) that contains more than $3d-4$ flat points for P. Then l is a flat line for P.*

Proof The line l contains more than $3d-4$ flat points for P, hence for each $i=1,2,3$, $\Pi_i(P)$ vanishes at more than $\deg(\Pi_i(P))$ points of l. Hence, using 3.1, we must have that $l \subset Z(\Pi_i(P))$. Since l is in general position, it means that all points on it are in general position. Using lemma 3.12, we obtain our conclusion. ∎

We are now ready to provide an upper bound for the number of flat lines for a given polynomial P which has no linear factors (no planes contained in the zero set). As for the case of critical lines, we do it first for irreducible polynomials.

Lemma 3.14 *An irreducible trivariate polynomial P of degree $d>1$ can have at most $3d^2-4d$ flat lines.*

Proof We note that if a is a flat point, then $\Pi_i(P)(a)=0$, for $i=1,2,3$. Assume for contradiction P has more than $3d^2-4d$ flat lines. Using Bézout's theorem 3.3 again and noticing that P is irreducible, we obtain that P is a factor of $\Pi_i(P)$, for all i. This implies that for all regular points of P in general position, the second-order Taylor approximation is a plane (with the possible exception of those points for which the tangent plane is parallel to one of the coordinate axes).

Let a be a regular point of P in general position. Using the Implicit Function Theorem, we can parameterise $Z(P)$ in a neighbourhood around a with a map $q:\mathbb{R}^2 \to Z(P)$, so we have

$$q(u,v)=(x(u,v),y(u,v),z(u,v))$$

for $(u,v)\in U$ some open set.

Hence we know $P((x(u,v),y(u,v),z(u,v))=0$, and differentiating with respect to u, we obtain

$$0=P_x x_u+P_y y_u+P_z z_u=\nabla P(\mathbf{x})\cdot \mathbf{x}_u$$

in a neighbourhood around a. Differentiating again with respect to u, we get

$$\begin{aligned}0&=P_x x_{uu}+P_y y_{uu}+P_z z_{uu}+(P_{xx}x_u+P_{xy}y_u+P_{xz}z_u)x_u+\\&+(P_{xy}x_u+P_{yy}y_u+P_{yz}z_u)y_u+(P_{xz}x_u+P_{yz}y_u+P_{zz}z_u)z_u=\\&=\nabla P(\mathbf{x})\cdot \mathbf{x}_{uu}+\mathbf{x}_u^T H_P(\mathbf{x})\mathbf{x}_u\end{aligned}$$

But $\mathbf{x}_u$ belongs to the tangent plane of $Z(P)$ at $\mathbf{x}$, so $\mathbf{x}_u^T H_P(\mathbf{x})\mathbf{x}_u=0$ because we know the second-order Taylor approximation vanishes on the tangent plane by assumption (a is in general position, i.e. π_a is not parallel to the coordinate axes, and we can find a neighbourhood around a such that for all points in the neighbourhood are in general position, so statement holds). Hence

$$\nabla P(\mathbf{x})\cdot \mathbf{x}_{uu}=0$$

Another way to write the derivative of $\nabla P(\mathbf{x})\cdot \mathbf{x}_u=0$ with respect to u is

$$0=(\nabla P(\mathbf{x}))_u\cdot \mathbf{x}_u+\nabla P(x)\cdot \mathbf{x}_{uu}=(\nabla P(\mathbf{x}))_u\cdot \mathbf{x}_u$$

Similarly, we get

$$(\nabla P(\mathbf{x}))_u\cdot \mathbf{x}_u=(\nabla P(\mathbf{x}))_u\cdot \mathbf{x}_v=(\nabla P(\mathbf{x}))_v\cdot \mathbf{x}_u=(\nabla P(\mathbf{x}))_v\cdot \mathbf{x}_v=0\ .$$

But $\mathbf{x}_u$ and $\mathbf{x}_v$ span the tangent space at $\mathbf{x}$, which means that $(\nabla P(\mathbf{x}))_u$ and $(\nabla P(\mathbf{x}))_v$ are perpendicular to the tangent space. Therefore the direction of $\nabla P(\mathbf{x})$ does not change, which means $Z(P)$ is locally a plane.

Using irreducibility of P, this implies that $Z(P)$ is plane, contradicting that the degree of P is greater than 1. ∎

Lemma 3.15 *A square-free trivariate polynomial P with no linear factors has at most $3d^2 - 4d$ flat lines.*

Proof We proceed by induction on the degree d. If P is irreducible, the claim follows from the previous proposition. Now suppose $P{=}P_1P_2$, where P_1 and P_2 are square-free polynomials with no common factors and none of them with linear factors. Say their degrees are d_1, d_2 respectively.

We would like to obtain a similar result as in the case of critical line, that is a flat line of P is either contained in $Z(P_1) \cap Z(P_2)$ or it is a flat line for at least one of P_1 or P_2. Suppose l is a flat line on which P_1 and P_2 don't vanish simultaneously. So without losing the generality, let $a \in l$ a regular point in general position so that $P_1(a) = 0$ and $P_2(a) \neq 0$ (note that $Z(P_2) \cap l \leq d_2$)

We have

$$0 \neq \nabla P(a) = P_1(a)\nabla P_2(a) + \nabla P_1(a)P_2(a) = \nabla P_1(a)P_2(a)$$

Observe that

$$H_P(x) = P_1(x)H_{P_2}(x) + (\nabla P_1(x))\,(\nabla P_2(x))^T + (\nabla P_2(x))(\nabla P_1(x))^T + P_2(x)H_{P_1}(x)$$

Hence for all $u \in \mathbb{R}^3$, we get

$$u^T H_P(a)u = P_2(a)u^T H_{P_1}(a)u + 2(u \cdot \nabla P_1(a))(u \cdot \nabla P_2(a))$$

Let

$$u_i = \nabla P(a) \times e_i = P_2(a)(\nabla P_1(a) \times e_i)$$

for $i = 1, 2, 3$, and by plugging in the equation above, we obtain

$$\Pi_i(P)(a) = P_2(a)^3\ \Pi_i(P_1)(a)\ .$$

Since a is flat for P, we have that $\Pi_i(P)(a) = 0$, for all i, hence $\Pi_i(P_1)(a) = 0$, which means that a is flat for P_1.

We observe that there are at most d_1d_2 lines in $Z(P_1) \cap Z(P_2)$, as they have no common factor (again using Bézout). By induction hypothesis, P_1 has at most $3d_1^2 - 4d_1$ flat lines, and similar estimate holds for P_2. Therefore, the total number of critical lines is at most

$$d_1d_2 + 3d_1^2 - 4d_1 + 3d_2^2 - 4d_2 \leq 3d^2 - 4d\ . \qquad \blacksquare$$

3.3 Ruled surfaces

Let $P \in \mathbb{R}[x, y, z]$ be a trivariate polynomial of degree d. We say that the surface $Z(P)$ is *ruled* if it contains a line passing through every point. In this section we will discuss some geometric properties of the ruled surfaces which will turn out to be essential in providing a bound for number of intersections between lines in the setting of the first section.

Examples of ruled surfaces are planes, cylinders or cones. A ruled surface Z can be informally described as the set of points swept by a "moving line". Formally, a ruled surface has locally a parametrization $g : U \to Z$ such that

$$g(u, v) = \alpha(u) + v\beta(u)\ , \tag{7}$$

where α and β are curves in $\mathbb{R}^3$, for some $U \subset \mathbb{R}^2$ (this means that Z is obtained by sweeping a line along 2 curves α and β). The set of such lines is called a *ruling*. Hence a ruling is a set $\mathscr{L}$ of lines in Z such that each point in Z belongs to exactly one line in $\mathscr{L}$.

First, we are going to show that if a surface $Z(P)$ contains too "many" lines, then it must have a ruled factor.

Recall the polynomials $P_1, P_2, \dots P_d$, where P_j is the j-th order term of the Taylor expansion (for $1 \leq j \leq d$). Using Lemma 3.6, we see that the surface $Z(P)$ is ruled if and only for every $a \in Z(P)$, there exists some direction $\mathbf{v}$ such that

$$P_1(a, \mathbf{v}) = P_2(a, \mathbf{v}) = \cdots = P_d(a, \mathbf{v}) = 0 \ .$$

We want to provide a weaker condition which characterizes ruled surfaces. Given a trivariate polynomial P of degree $d \geq 3$, we say a point $a \in Z(P)$ is a *flecnode* if there is a line passing through it which agrees with the surface up to order 3, i.e. there exists direction $\mathbf{v}$ such that

$$P(a) = P_1(a, \mathbf{v}) = P_2(a, \mathbf{v}) = P_3(a, \mathbf{v}) = 0 \tag{8}$$

We would like to give an equivalent condition of flecnode points which does not depend on the direction $\mathbf{v} = (v_1, v_2, v_3)$. We give a sketch proof that there exists a trivariate polynomial which vanishes on flecnode points. Assume that a is a flecnode point which is not critical ($\nabla P(a) \neq \mathbf{0}$), so suppose without losing the generality that $P_x(a) \neq 0$. The condition $P_1(a, \mathbf{v}) = 0$ can be rewritten as $P_x(a)v_1 + P_y(a)v_2 + P_z(a)v_3 = 0$, so we can write v_1 as a linear combination of v_2 and v_3. Hence the equations $P_2(a, \mathbf{v}) = 0$ and $P_3(a, \mathbf{v}) = 0$ can be written as homogeneous equations in v_2 and v_3 of degree 2 and 3 respectively. From the equation of degree 2 we get 2 possibilities for v_3 as a linear equation of v_2 (i.e. v_3 will be v_2 multiplied by a rational polynomial which depends only on derivatives up so second order of P evaluated at a). Plugging these 2 values into the last equations, we obtain a polynomial $Fl(P) \in \mathbb{R}[x, y, z]$ such that $v_3^3 Fl(P)(a) = 0$. Indeed, this polynomial is zero at $a \in Z(P)$ if and only if there are v_1, v_2 and v_3 such that $P_1(a, \mathbf{v}) = P_2(a, \mathbf{v}) = P_3(a, \mathbf{v}) = 0$.

$Fl(P)$ is called the *flecnode polynomial* of P. It is not hard to check (but tedious!) that the degree of $Fl(p)$ is $11d - 24$.

The following result is one of the most important results in the theory of ruled surfaces.

Proposition 3.16 *Let P be trivariate polynomial. Then the surface $Z(P)$ is ruled if and only of $Fl(P)$ vanishes on $Z(P)$.*

The proof of this statement is beyond the scope of this project. A rigorous proof can be found in [12]. However, we can remark that one direction is trivial, since if the surface is ruled, there is a line contained in the surface at every point, so it clearly agrees up to order 3. The other direction is the more interesting one, which says that it is enough to check that at each point, a line is contained up to order 3.

The following corollary is important for our purpose.

Corollary 3.17 *Let P be a trivariate polynomial of degree d. Suppose that $Z(P)$ contains more than $11d^2 - 24d$ lines. Then P has a ruled factor.*

Proof We note that $Fl(P)$ an P vanish simultaneously on more than $11d^2 - 24d$ lines, so by using Bézout's Therem 3.3, they must have a common factor Q. We claim that $Z(Q)$ is ruled.

Indeed, for every regular point $a \in Z(Q)$, we have $Fl(P)(a) = 0$, hence there exists a line which agrees with the surface $Z(Q) \subset Z(P)$ up to order 3. Hence $Fl(Q)(a) = 0$, so by the previous proposition $Z(Q)$ is ruled. ∎

Next we turn attention to some particular types of ruled surfaces. A *doubly-ruled* surface Z is a surface in $\mathbb{R}^3$ such that any point in Z lies in at least 2 lines contained in the surface. An example of a doubly-ruled surface is a *regulus*. A regulus is defined as the union of all the lines that intersect 3 given pairwise-skew lines l_1, l_2 and l_3. We will next show that reguli are irreducible doubly-ruled surfaces of degree 2, and later in the section show that all irreducible ruled surfaces must be either planes, a reguli or singly-ruled.

Lemma 3.18 *A regulus is a doubly-ruled irreducible surface of degree 2.*

Proof Let R be a regulus and l_1, l_2 and l_3 the three pairwise-skew lines from the definition of R. We will use an argument similar as in the proof of lemma 3.5. Choose arbitrarily 3 points on each of the lines l_1, l_2, l_3. There are 9 points in total, and as $\binom{3+2}{2} = 10 > 9$, then using lemma 3.4 there exists a non-zero polynomial P of degree at most 2 which vanishes on all these 9 points. This means that P vanishes on at least 3 points of l_i, for all $1 \leq i \leq 3$, hence using lemma 3.1, we obtain that $l_i \subset Z(P)$, for all i. Hence P has degree 2 and is irreducible, since a plane contains at most one of the lines l_1, l_2, l_3.

Now let l be a line that intersects l_1, l_2 and l_3. Since the points of intersection must be distinct, it follows that l has at least 3 intersections with $Z(P)$, hence $l \subset Z(P)$, so $R \subset Z(P)$.

For the other direction, let $a \in Z(P) \setminus (\pi_{12} \cup \pi_{13} \cup \pi_{23})$, where π_{12} is the plane containing l_1 parallel to l_2 and π_{23}, π_{13} defined similarly. Then there exists a line passing through a and intersecting l_1 and l_2, call it l_{12}. So l_{12} has at least 3 intersections with $Z(P)$, so $l_{12} \subset Z(P)$. Similarly, we find $l_{13}, l_{23} \subset Z(P)$. Note that if $l_{12} \equiv l_{13} \equiv l_{23}$, then we have found a line passing through a which intersects l_1, l_2 and l_3, hence $a \in R$. Otherwise, there exist 3 lines in $Z(P)$ intersecting at a, hence a is flat. Recall that in section 3.2 we showed that if all points of $Z(P)$ in a neighbourhood of a are flat, then $Z(P)$ mush have a planar component, which is a contradiction. This means we can find a sequence of points $a_n \to a$ such that $a_n \in R$, which by continuity implies that $a \in R$. So we've shown that almost all points in $Z(P)$ belong to R (with the possible exception of 3 curves). Hence the polynomial P is unique.

To see that R is doubly-ruled, it is enough to find two disjoint rulings for R. One ruling $\mathscr{L}$ can be defined as the set of lines which intersect the given lines l_1, l_2 and l_3. Now let $l'_1, l'_2, l'_3 \in \mathscr{L}$ be 3 pairwise-skew lines. Let R' be the regulus defined by the union of lines which intersect each of the lines l'_1, l'_2, l'_3. It is easy to see that $R' = R$, because if $l' \subset R'$, is a line, P vanishes on at least 3 points of l' (the intersections with l'_i), so $l' \subset Z(P)$. We now just take the second ruling $\mathscr{L}'$ to be the set of lines which intersect l'_1, l'_2 and l'_3. ∎

Remark Note that the proof above implies that all doubly-ruled irreducible surfaces of degree 2 are reguli, since we can find 3 pairwise-skew lines in $Z(P)$ and use them to define the corresponding regulus.

We now turn out attention to some geometrical aspects of surfaces. We call a point $a \in Z$ *exceptional* if there are infinitely many lines in the surface that pass through a. For example, every point in the plane is exceptional, and the apex is the only exceptional point of a cone.

Lemma 3.19 *Let P be an irreducible trivariate polynomial and $a \in Z(P)$ an exceptional point. Then for any other point $x \in Z(P)$, the line which passes through a and x is contained in $Z(P)$.*

Proof Let $x \in Z(P)$. To check whether the line through a and x is contained in $Z(P)$, it is enough to check that

$$P_1(x, x-a) = P_2(x, x-a) = \cdots = P_d(x, x-a) = 0$$

where d is the degree of P. As a is fixed, $Q_j(x) := P_j(x, x-a)$ is a trivariate polynomial, for all $1 \leq j \leq d$. If l is a line that passes through a contained in $Z(P)$, then all Q_j vanish on l.

By assumption, a is exceptional, so P and Q_j vanish on infinitely many lines. Using that P is irreducible and Bézout theorem 3.3, we get that P is a factor of Q_j, for all j. So Q_j vanish on all $Z(P)$, for all j, which implies the claim. ∎

Next, we want to define a similar concept to exceptional points, but for lines. We call a line $l \subset Z(P)$ *exceptional* if there are infinitely many lines in $Z(P)$ which intersect l at non-exceptional points. This is equivalent of saying that l contains infinitely many non-exceptional points that belong a different line in $Z(P)$. Next we prove a lemma similar to the one for exceptional points.

Lemma 3.20 *Let P be an irreducible trivariate polynomial of degree $d \geq 2$. Then there exists an algebraic curve C so that for all $x \in Z(P) \setminus C$, there is a line in $Z(P)$ passing through x and intersecting l.*

Proof To make the computations easier, suppose without losing the generality that l is the x-axis (we can achieve this by a change of coordinates). We are going to show that $C = Z(P) \cap Z(P_x)$ works. In order to check that C is indeed an algebraic curve, we have to make sure that P and P_x have no common factor. As P is irreducible, the only way this could happen is if $P_x \equiv 0$, which means P depends only on the variables y and z. So $Z(P)$ is invariant under translation in the x-direction. Let l' be any line in $Z(P)$ intersecting l, so $Z(P)$ contains all translations of l', so it has a plane as a factor, contradiction.

Now let a be a point in $Z(P)$ such that $P_x(a) \neq 0$. In particular, $\nabla P(a) \neq 0$. We would like to find a line in $Z(P)$ passing through a and intersecting l. We note that any line contained in $Z(P)$ that passes through a must be contained in the tangent plane of $Z(P)$ at a, called π_a (this easy fact is proved in 3.10). Say $a = (a_1, a_2, a_3)$. Then $\pi_a \cap l = (a', 0, 0)$, where a' is such that

$$P_x(a)(a_1 - a') + P_y(a)a_2 + P_z(a)a_3 = 0 \ .$$

Hence

$$a' = \frac{P_x(a)a_1 + P_y(a)a_2 + P_z(a)a_3}{P_x(a)} \ .$$

Therefore, we want to show that for all $\mathbf{x} = (x, y, z) \in Z(P)$ such that $P_x(\mathbf{x}) \neq 0$, the line between $\mathbf{x}$ and

$$\mathbf{x}' = \left(\frac{P_x(\mathbf{x})x + P_y(\mathbf{x})y + P_z(\mathbf{x})z}{P_x(\mathbf{x})}, 0, 0 \right)$$

to be contained in $Z(P)$. This is equivalent to

$$P_1(\mathbf{x}, \mathbf{x} - \mathbf{x}') = P_2(\mathbf{x}, \mathbf{x} - \mathbf{x}') = \cdots = P_d(\mathbf{x}, \mathbf{x} - \mathbf{x}') = 0 \ .$$

Note that for all $1 \leq t \leq d$, we have

$$P_t(\mathbf{x} - \mathbf{x}') = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=t}} \frac{1}{i!j!k!} \left(x - \frac{P_x(\mathbf{x})x + P_y(\mathbf{x})y + P_z(\mathbf{x})z}{P_x(\mathbf{x})} \right)^i y^j z^k \frac{\partial^{i+j+k} P}{\partial x^i \partial y^j \partial z^k}(\mathbf{x}) \ .$$

We can view this as solving for d rational functions of 3 variables which only have a power of $P_x(\mathbf{x})$ in their denominators. Denote by $Q_j(\mathbf{x}) = (P_x(\mathbf{x}))^j P_j(\mathbf{x}, \mathbf{x} - \mathbf{x}')$, so we need to show that

$$Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = \cdots = Q_d(\mathbf{x}) = 0$$

for all $\mathbf{x} \in Z(P)$ such that $P_x(\mathbf{x}) \neq 0$. But we know l is an exceptional line, hence Q_j and P have infinitely many lines in common, so using Bézout's theorem again, we obtain that P is a factor of each Q_j, which concludes the claim.

The following result will just wrap together the last 2 lemmas into a key result:

Lemma 3.21 *Let P be an irreducible triariate polynomial such that $Z(P)$ is neither a plane or a regulus. Then S has at most one exceptional point and at most two exceptional lines.*

Proof Suppose a_1 and a_2 are two exceptional points of $Z(P)$. Let any point $\mathbf{x} \in Z(P)$ not contained on the line between a_1 and a_2. Then by Lemma 3.19, $\mathbf{x}$ will be contained in at least two lines of $Z(P)$, which is a contradiction with $Z(P)$ not doubly-ruled.

Now, suppose l_1, l_2 and l_3 are three distinct exceptional lines of $Z(P)$ with the corresponding curves C_1, C_2, C_3 given by the Lemma 3.20. Assuming $Z(P)$ is not doubly-ruled, the generic point $\mathbf{x} \in Z(P)$ will be contained in at most one line of $Z(P)$. Now let $\mathbf{x} \in Z(P) \setminus (C_1 \cup C_2 \cup C_3)$. Then by the previous lemma we must have that there is a line passing through $\mathbf{x}$ that intersects each of l_1, l_2 and l_3. If any of the l_i's are coplanar, it means that $Z(P)$ contains infinitely many lines in some plane, so applying again Bézout's theorem, we derive a contradiction. Similarly, if l_1, l_2 and l_3 are pairwise skew, then $Z(P)$ will contain infinitely many lines of a regulus, so we derive again a contradiction. ∎

Remark Note that this implies that if P irreducible and $Z(P)$ ruled, then $Z(P)$ is either a plane, a regulus or it is singly-ruled.

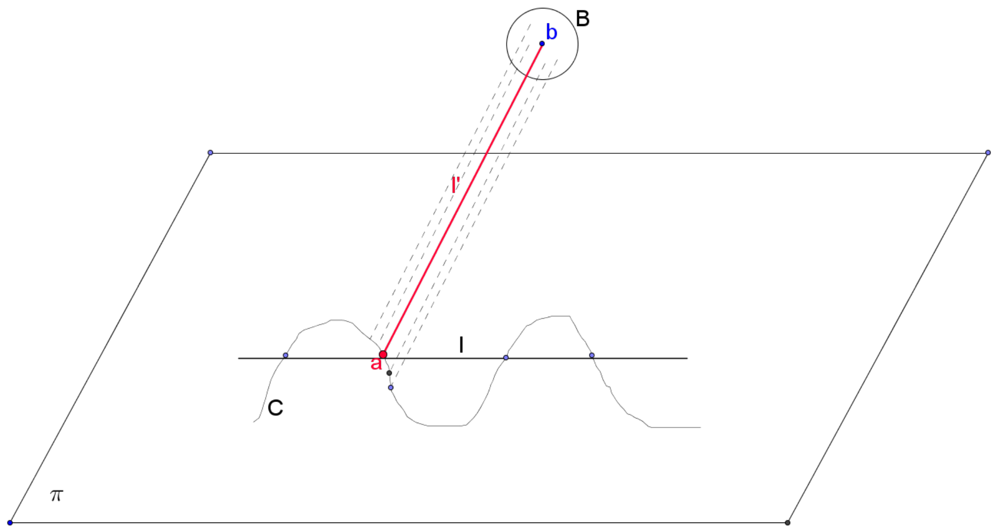
It is not hard to check that if we have N lines in a plane or in a regulus, it is possible to have $\sim N^2$ points of intersection. After all this build-up, we are ready to state and prove the main theorem of this chapter. We will show that planes and reguli are the only surfaces with "many" intersections between lines.

Theorem 3.22 *Let P be a trivariate polynomial of degree at most d such that $Z(P)$ is ruled and has no factors which are planes or reguli. Let L be a set of lines in $Z(P)$ such that $|L| \leq Cd^2$, for some constant $C \geq 1$. Then the set of intersections of lines of L has size at most $4Cd^3$.*

Proof Say $P = P_1 P_2 \dots P_k$, where P_i is irreducible for each $1 \leq i \leq k$. First we notice that a point is exceptional for the surface $Z(P)$ if and only if it is exceptional for at least one of the $Z(P_i)$ (indeed, a point is exceptional if there are infinitely many lines in $Z(P)$ passing through it, so there must be infinitely many lines in one of the irreducible components). Similarly, a line is exceptional for $Z(P)$ if and only of it is exceptional for at least on of the $Z(P_i)$. Hence, using the previous lemma, we obtain there are at most d exceptional points and $2d$ exceptional lines. So there are at most $2Cd^3$ intersections between exceptional lines and lines of L.

We now focus on intersections of non-exceptional lines in L at non-exceptional points. Fix l a non-exceptional line. Let π be any plane containing l. So $Z(P) \cap \pi$ is an algebraic curve of degree at most d (as $Z(P)$ is plane-free). Clearly, l is contained in this algebraic curve. The other component is a curve C of degree at most $d-1$. So $Z(P) \cap \pi = l \cup C$. Using Bézout's theorem, it is easy to observe the $|l \cap C| \leq d-1$, otherwise l is contained in C. We would like to show that if $l' \subset Z(P)$ is any other non-exceptional line and $l \cap l' = \{a\}$, where a is non-exceptional, then $a \in l \cap C$. This would imply that l has at most $d-1$ intersections with other non-exceptional lines at non-exceptional points, so there are at most Cd^3 intersections between non-exceptional lines of L at non-exceptional points.

In order to prove the claim, fix a point non-exceptional point $b \in l'$ different from a. Fix a small neighbourhood B around b in $Z(P)$. We can choose this neighbourhood such that there is no exceptional point inside it and can by parameterised as in (7) such that all lines from the ruling intersect π. So these lines must intersect $l \cup C$ in a neighbourhood of a. As a is non-exceptional, only finitely many of these lines pass through a. Since the set of exceptional points is finite, we can also arrange such that the intersections of these lines with π contain no exceptional point. Since l is non-exceptional, only finitely many lines in the ruling can intersect l. Hence all but finitely many must intersect C. By continuity, this is possible if and only if $a \in C$.



Putting everything together, we have at most $2Cd^3 + Cd^2 + d \leq 4Cd^3$ intersections between lines of L. ∎

4 Polynomial cell decomposition

The goal of this section is to prove that, given a finite set of points in $\mathbb{R}^n$, we can construct a cell decomposition of $\mathbb{R}^n$ such that the walls of the cells are contained in the zero set of a polynomial of "low" degree and that each cell contains "few" of the original points. Of course, in the following pages, everything will be treated carefully and we will obtain some additional important results. We begin by recalling the classical Borsuk-Ulam and Ham sandwich theorems.

Theorem 4.1 (Borsuk-Ulam) *Let $T : S^n \to \mathbb{R}^n$ be an odd continuous map ($T(x) = -T(-x)$ for all $x \in \mathbb{R}^n$) . Then there exist $x_0 \in S_n$ such that $T(x_0) = 0$.*

Theorem 4.2 (Ham sandwich theorem) *Let $B_1, B_2, \ldots, B_n$ be bounded open measurable subsets of $\mathbb{R}^n$. Then there exists an hyperplane which bisects each B_i (i.e. there exists $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$ such that $B_i \cap \{(x_1, ..., x_n) \in \mathbb{R}^n : a_0 + a_1x_1 + ... + a_nx_n > 0\}$ and $B_i \cap \{(x_1, ..., x_n) \in \mathbb{R}^n : a_0 + a_1x_1 + ... + a_nx_n < 0\}$ have the same Lebesque measure).*

We would like to obtain a generalisation of the Ham sandwich theorem such that each set is not bisected by a plane (a degree 1 polynomial), but by a polynomial of greater degree.

Theorem 4.3 (Polynomial ham sandwich theorem) *Let $d \geq 1$ be an integer and B_1, $B_2, \ldots B_m$ be bounded open sets in $\mathbb{R}^n$, with $m < \binom{n+d}{n}$. Then there exists a non-trivial polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most d such that the algebraic set $Z(P)$ bisects each B_i (i.e. $B_i \cap \{x \in \mathbb{R}^n : P(x) > 0\}$ and $B_i \cap \{x \in \mathbb{R}^n : P(x) < 0\}$ have the same Lebesque measure).*

Proof Let V be the vector space of polynomials $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most d, which has dimension $\binom{n+d}{d}$ over $\mathbb{R}$. Consider the map $T : V \to \mathbb{R}^m$ defined by

$$T(P) := \left(\int_{B_i} sign(P) \right)_{i=1}^{m}$$

which is continuous and odd. We can view this as a map $T : \mathbb{R}^{\binom{n+d}{d}} \to \mathbb{R}^m$. We can now apply Borsuk-Ulam theorem (just restrict the map to S^m as $m < \binom{n+d}{d}$) and see that $T(P) = 0$, for some non-zero $P \in V$, so the claim follows. ∎

We now want to adapt the previous theorem to finite sets of points instead of bounded measurable sets.

Lemma 4.4 (Polynomial ham sandwich theorem, discrete case) *Let $d \geq 1$ an integer and let $S_1, S_2, \ldots, S_m$ be finite sets of points in $\mathbb{R}^n$, with $m < \binom{n+d}{d}$. Then there exists a non-trivial polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most d such that the $Z(P)$ bisects each B_i, in the sense that $\{x \in S_i : P(x) > 0\}$ and $\{x \in S_i : P(x) < 0\}$ both have cardinality at most $|S_i|/2$, for all i.*

Proof Let V be the vector space of polynomials $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most d. For any $\delta > 0$, we define $B_{i,\delta}$ to be the union of the δ-balls centered at the points of S_i. Clearly, these are open bounded subsets of $\mathbb{R}^n$. Therefore we can apply Theorem 4.3 to obtain a non-zero polynomial $P_\delta \in V$ that bisects each $B_{i,\delta}$.

On V, we look at the sup norm $\|P\|$ given by the maximal absolute value of the coefficients of P. We can assume without losing the generality that $\|P_\delta\| = 1$, as scaling preserves the zero set of a polynomial. As V is finite dimensional, we can find a sequence $\delta_m \to 0$ such that P_{δ_m} converges in $(V, \|\cdot\|)$. As V is closed, let P be the limit polynomial and note that $\|P\| = 1$. In particular, we note that $P_{\delta_m} \to P$ pointwise, and therefore converges uniformly on compact sets.

We claim that P bisects each set S_i. Suppose for contradiction $|\{x \in S_i : P(x) > 0\}| > |S_i|/2$, for some i (the case $P < 0$ is similar). Let $S_i^+ = \{x \in S_i : P(x) > 0\}$. Clearly, we can find

$\epsilon > 0$ such that $P > \epsilon$ on all ϵ-balls around each point of S_i^+, as P is continuous and S_i^+ finite. Since P_{δ_m} converges uniformly on compact sets, we can find N large enough such that $\delta_N < \epsilon$ and $P_{\delta_N} > 0$ on the ϵ-balls around points of S_i^+. But this means that $P_{\delta_N} > 0$ on more than half of B_{i,δ_N}, contradiction. ∎

Remark Using the crude inequality

$$\binom{n+d}{d} > \frac{d^n}{n!} > \frac{d^n}{n^n}$$

we observe that the previous lemma implies that some finite sets $S_1, \ldots, S_m$ can be bisected by a polynomial of degree at most d if $d^n \geq n^n m$, hence those sets can be bisected by a polynomial of degree at most $2nm^{1/n}$.

We are now ready to prove the main result of this section.

Theorem 4.5 (Cell decomposition) *Let $d \geq 1$ and let S be a finite set of points in $\mathbb{R}^n$. Then there exists a non-trivial polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $O(d^{1/n})$ and a partition*

$$\mathbb{R}^n = Z(P) \cup \Omega_1 \cup \cdots \cup \Omega_M$$

with $M = O(d)$ such that each Ω_i in an open set with boundary contained in $Z(P)$ and

$$|S \cap \Omega_i| \leq \frac{|S|}{M} \ ,$$

for all i.

Proof We will prove that when M is a power of 2, then we can find a non-trivial polynomial of degree $\leq CM^{1/n}$ (for some universal constant C we will define later in the proof) and cells $\Omega_1, \ldots, \Omega_M$ with boundary contained in $Z(P)$ such that $|S \cap \Omega_i| \leq \frac{|S|}{M}$, for all i. This clearly implies the lemma, as we just choose M to be the least power of 2 greater than d.

Say $M = 2^k$. Using the remark above, we find a polynomial P_1 of degree $d_1 \leq 2n$ such that the finite sets $S_1 = \{x \in \mathrm{R}^{\mathrm{n}} : P_1(x) > 0\} \cap S$ and $S_{-1} = \{x \in \mathrm{R}^{\mathrm{n}} : P_1(x) < 0\} \cap S$ have cardinality at most $\frac{|S|}{2}$. Similarly, we find polynomial P_2 of degree $d_2 \leq 2n \cdot 2^{1/n}$ that bisects S_1 and S_{-1}, hence the finite sets $S_{(1,1)} = S_1 \cap \{P_2 > 0\}$, $S_{(1,-1)} = S_1 \cap \{P_2 < 0\}$, $S_{(-1,1)} = S_{-1} \cap \{P_2 > 0\}$ and $S_{(-1,-1)} = S_{-1} \cap \{P_2 < 0\}$ have cardinality at most $\frac{|S|}{4}$. By iterating the argument, for all $j \leq k$, we find a polynomial P_j of degree at most $2n \cdot 2^{(j-1)/n}$ such that for all $\epsilon \in \{1, -1\}^{j-1}$, the sets defined by $S_{(\epsilon,1)} = S_\epsilon \cap \{P_j > 0\}$ and $S_{(\epsilon,-1)} = S_\epsilon \cap \{P_j < 0\}$ have cardinality at most $|S|/2^j$.

Now, for all $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in \{1, -1\}^k$, we define the cells

$$\Omega_e = \{\epsilon_1 P_1 > 0\} \cap \{\epsilon_2 P_2 > 0\} \cap \cdots \cap \{\epsilon_k P_k > 0\}$$

Then clearly all Ω_ϵ are disjoint, $S \cap \Omega_\epsilon = S_\epsilon$ and if we set $P = P_1 P_2 \ldots P_k$, then

$$\mathrm{R}^{\mathrm{n}} = Z(P) \cup \bigcup_{\epsilon \in \{1,-1\}^k} \Omega_\epsilon$$

Now,

$$\deg(P) = \sum_{j=1}^{k} \deg(P_j) \leq 2n \sum_{j=1}^{k} 2^{(j-1)/n} \leq \frac{2n}{2^{1/n} - 1} 2^{k/n} = \frac{2n}{2^{1/n} - 1} M^{1/n}$$

Take $C = \frac{2n}{2^{1/n}-1}$ and the claim follows. ∎

Remark The cells Ω_i do not have to be connected. Actually, each cell is the union of some connected components of $\mathbb{R}^n \setminus Z(P)$.

If we want to give explicit constants, we can reformulate the theorem as follows:

Theorem 4.6 *Let $d \geq 1$ and let S be a finite set of points in $\mathbb{R}^n$. Then there exists and a partition*

$$\mathbb{R}^n = Z(P) \cup \Omega_1 \cup \cdots \cup \Omega_M$$

with $M \leq 2d$ and non-trivial polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most $\frac{2n \cdot 2^{1/n}}{2^{1/n}-1} d^{1/n}$such that each Ω_i in an open set with boundary contained in $Z(P)$ and

$$|S \cap \Omega_i| \leq \frac{|S|}{d} \ ,$$

for all i.

We end this section with an application of polynomial cell decomposition. We will prove the Szeméredi - Trotter theorem, one of the fundamental results in incidence geometry.

Theorem 4.7 (Szeméredi-Trotter) *Let P be a finite set of points in and L a finite set of lines $\mathbb{R}^2$. Let $I(P, L) := \{(p, l) \in P \times L : p \in L\}$. Then*

$$|I(P, L)| = O(|P|^{2/3}|L|^{2/3} + |P| + |L|) \ .$$

We first prove an easy lemma which provides a worse bound, but is needed for the proof of the Szeméredi - Trotter theorem.

Lemma 4.8 *For any finite set of points P and finite set of lines L in $\mathbb{R}^2$, we have*

$$|I(P, L)| \leq |P||L|^{1/2} + |L| \ .$$

Proof Denote by $d(l)$ the number of points on the line l. Then clearly $|I(P, L)| = \sum_{l \in L} d(l)$.

Using Cauchy-Schwarz, we obtain

$$\left(\sum_{l \in L} d(l)^2\right) |L| \geq |I(P, L)|^2 \ .$$

We observe that $\sum_{l \in L} d(l)^2$ is the number of triples $(p_1, p_2, l) \in P \times P \times L$ such that $p_1 \in l$ and $p_2 \in l$. If $p_1 \neq p_2$, then there is at most one line in L which contains p_1 and p_2. Therefore we obtain

$$\sum_{l \in L} d(l)^2 \leq |P|^2 + |I(P, L)| \ .$$

Therefore,

$$|I(P, L)|^2 - |L||I(P, L)| - |L||P|^2 \leq 0 \ .$$

Hence

$$|I(P, L)| \leq \frac{|L| + \sqrt{|L|^2 + 4|L||P|^2}}{2} \leq |L| + |P||L|^{1/2} \ . \quad \blacksquare$$

Proof (Szeméredi-Trotter) We apply theorem 4.5 for some parameter d. Then there exists a non-trivial polynomial Q of degree at most $16d^{1/2}$ and a decomposition

$$\mathbb{R}^2 = Z(Q) \cup \Omega_1 \cup \cdots \cup \Omega_m$$

such that $m \leq 2d$ and each cell contains at most $|P|/d$ points of P. We can assume without losing the generality that Q is square-free (removing the repeated factors won't change the zero set).

We now can write

$$|I(P,L)| = |I(Z(Q) \cap P, L)| + \sum_{i=1}^{m} |I(P \cap \Omega_i, L)| \ . \tag{9}$$

Denote by L_i the set of lines in L which have non-empty intersection with Ω_i. It follows that $I(P \cap \Omega_i, L) = I(P \cap \Omega_i, L_i)$. Applying the previous lemma, we obtain

$$|I(P \cap \Omega_i, L_i)| \leq |P \cap \Omega_i||L_i|^{1/2} + |L_i| \leq \frac{|P|}{d}||L_i|^{1/2} + |L_i| \ . \tag{10}$$

As we've seen in the previous sections, every line not contained in $Z(Q)$ intersects $Z(Q)$ in at most $\deg(Q)$ points. This implies that each line not contained in $Z(Q)$ intersects at most $\deg(Q)+1$ of the cells in the decomposition, because when a line moves from one cell to another, it has to pass through $Z(Q)$. Therefore

$$\sum_{i=1}^{m} |L_i| = |\{(l, \Omega_i) : l \in L_i, 1 \leq i \leq m\}| \leq 16d^{1/2}|L|$$

(note that by plugging in the constant we obtain in 4.6, we can actually attain that the degree of Q is less than $15d^{1/2}$, so the inequality above clearly holds). By applying Cauchy-Schwarz, we get

$$\sum_{i=1}^{m} |L_i|^{1/2} \leq m^{1/2} \left(\sum_{i=1}^{m} |L_i| \right)^{1/2} \leq 4\sqrt{2}d^{3/4}|L|^{1/2} \ .$$

Putting this together with (9) and (10), we obtain

$$|I(P,L)| \leq |I(Z(Q) \cap P, L)| + 4\sqrt{2}d^{-1/4}|P||L|^{1/2} + 16d^{1/2}|L| \ . \tag{11}$$

Now we would like to bound $|I(Z(Q) \cap P, L)|$. Let L' be the subset of L of lines contained in $Z(Q)$ and $L'' = L \setminus L'$. Then each line in L'' intersects at most $16d^{1/2}$ times $Z(Q)$ (again by Bézout), therefore

$$|I(Z(Q) \cap P, L")| \leq 16d^{1/2}|L| \ .$$

Let P' be the subset of $Z(Q) \cap P$ of critical points (i.e. points p such that $\nabla Q(p) = 0$) and $P'' = (P \cap Z(Q)) \setminus P'$. Each point in P'' belongs to at most one line in $Z(Q)$ (the tangent line at that point), hence

$$|I(P", L')| \leq |P| \ .$$

It remains to bound $|I(P', L')|$. As Q is square-free, we observe that the components of ∇Q and Q have no common factors. Once again, Bézout theorem implies that each line in L' must intersect $Z(\nabla Q)$ at most $16d^{1/2}$ times (otherwise Q and ∇Q have a common line), therefore

$$|I(P', L')| \leq 16d^{1/2}|L| \ .$$

Hence,

$$|I(Z(Q) \cap P, L")| = |I(Z(Q) \cap P, L")| + |I(P", L')| + |I(P', L')| \leq 32d^{1/2}|L| + |P| \ . \tag{12}$$

Putting together (11) and (12), we obtain

$$|I(P,L)| \leq 4\sqrt{2}d^{-1/4}|P||L|^{1/2} + 48d^{1/2}|L| + |P| \ . \tag{13}$$

All is left to do is find a suitable value for the parameter d. Remember we must have $d \geq 1$ in order to apply theorem 4.5.

- If $|P|^2 \geq |L|$, set $d = P^{4/3}L^{-2/3}$ to obtain

$$|I(P,L)| \leq 4\sqrt{2}|P|^{2/3}|L|^{2/3} + 48d^{1/2}|L| + |P| \leq 56|P|^{2/3}|L|^{2/3} + |P| \ .$$

- Otherwise set $d = 1$ to obtain $|I(P,L)| \leq 56|L| + |P|$.

Therefore, putting everything together,

$$|I(P,L)| \leq 56(|P|^{2/3}|L|^{2/3} + d^{1/2}|L| + |P|) \ . \quad \blacksquare$$

Remark Note that we proved the result in $\mathbb{R}^2$, but the same result will hold in $\mathbb{R}^{\mathrm{n}}$ by just taking a generic projection from $\mathbb{R}^{\mathrm{n}}$ to $\mathbb{R}^2$ such that distinct lines are sent to distinct lines and distinct points to distinct points.

There is another form of the Szeméredi-Trotter which follows easily from Theorem 4.7 and which will be useful later.

Theorem 4.9 *Let L a finite set of lines $\mathbb{R}^2$. Then the number of points which belong to at least k lines ($k \geq 2$) is*

$$O\left(\frac{|L|^2}{k^3} + \frac{|L|}{k}\right) \ .$$

Proof Let S denote the number of points belonging to at least k lines. Denote $C \geq 1$ a constant for which Theorem 4.7 holds, i.e. for any finite set P' of points and L' finite set of lines in $\mathbb{R}^{\mathrm{n}}$, we have

$$|I(P',L')| \leq C(|P'|^{2/3}|L'|^{2/3} + |P'| + |L'|) \ .$$

If $k \leq 2C$, then there are at most $|L|^2$ intersections between lines of L, so

$$S \leq |L|^2 \leq 8C^3\left(\frac{|L|^2}{k^3}\right) \ .$$

Otherwise, denote P the set of points which belong to at least k lines and observe that $|P| = S$ and that $|I(P,L)| \geq kS$. Hence Theorem 4.7 implies that

$$kS \leq C(S^{2/3}|L|^{2/3} + S + |L|) \ .$$

So

$$kS \leq 2C(S^{2/3}|L|^{2/3} + |L|) \ ,$$

as $kS > 2CS$. Hence at least one of the two following possibilities must hold:

- $kS \leq 4CS^{2/3}|L|^{2/3}$, which is equivalent to

$$S \leq 64C^3\frac{|L|^2}{k^3} \ .$$

- $kS \leq 4C|L|$, which is equivalent to $S \leq 4C\frac{|L|}{k}$.

Hence we obtain

$$S \leq 64C^3\left(\frac{|L|^2}{k^3} + \frac{|L|}{k}\right) \ . \quad \blacksquare$$

Remark Looking back at the proof of Szémeredi- Trotter theorem, we can choose $C = 64 = 2^6$, we we obtain

$$S \leq 2^{24}\left(\frac{|L|^2}{k^3} + \frac{|L|}{k}\right) \ .$$

5 Proof for $k = 2$

In this section we will prove Theorem 2.11. Recall the setting we have obtained in section 2: we have a set $\mathscr{P} \subset \mathbb{R}^2$ of N points in plane, and we define the set of lines $\mathscr{L} = \{L_{pq} : p, q \in \mathscr{P}\}$, where L_{pq} is given by the parameterisation:

$$L_{pq} = \left\{ \left(\frac{p_x + q_x}{2}, \frac{p_y + q_y}{2}, 0 \right) + t \left(\frac{q_y - p_y}{2}, \frac{p_x - q_x}{2}, 1 \right) : t \in \mathbb{R} \right\} .$$

We have shown that any plane contains at most N lines of $\mathscr{L}$. We would like also to show that there are at most $O(N)$ lines of $\mathscr{L}$ in a regulus. Recall that for a point $a \in \mathbb{R}^2$ we have defined $\mathscr{L}_a = \{L_{ap} : p \in \mathbb{R}^2\}$ and we've shown in lemma 2.9 that every point in $\mathbb{R}^3$ belongs to exactly one line in $\mathscr{L}_a$. Also, we have seen that we can find a vector field

$$V = (V_1(x, y, z), V_2(x, y, z), V_3(x, y, z))$$

which is tangent to the unique line in $\mathscr{L}_a$ at every point in $\mathbb{R}^3$ and each component is a polynomial of degree 2. We will first prove the following lemma:

Lemma 5.1 *Let R be a regulus that contains at least 7 lines in $\mathscr{L}_a$. Then there is one ruling of R which is a subset of $\mathscr{L}_a$ (this means that for each point in R there exists a line in R passing through it which also belongs to $\mathscr{L}_a$).*

Proof Let P be an irreducible polynomial of degree 2 which defines R. Let L_{ap} be a line contained in R and $\mathbf{v}$ the direction of L_{ap}. For each point b on L_{ap}, we have that $V(b)$ has the same direction as L_{ap}, hence we obtain

$$0 = \lim_{h \to 0} \frac{P(b + h\mathbf{v}) - P(b)}{h} = \nabla_{\mathbf{v}} P(b) = V(b) \cdot \nabla P(b) .$$

This means that $V \cdot \nabla P$ vanishes on L_{ap}. As P is a polynomial of degree 2 and each component of V also has degree 2, we get that $V \cdot \nabla P$ has degree at most 3. Now R contains at least 7 lines of $\mathscr{L}_a$, which means that P and $V \cdot \nabla P$ simultaneously vanish on 7 lines. Noticing that P is irreducible of degree 2 and using Bézout's theorem 3.3, we obtain that P is a factor of $V \cdot \nabla P$. Therefore $V \cdot \nabla P$ vanishes on R, hence V is tangent to R at all points of R. So we obtain that for all points $x \in R$, the unique line in $\mathscr{L}_a$ which passes through x is tangent to R, hence we obtain a ruling of R consisting of lines in $\mathscr{L}_a$. ∎

Now we are ready to prove the set of lines $\mathscr{L} = \{L_{pq}, p, q \in P\}$ contains not many lines in a regulus.

Lemma 5.2 *There are at most $8N$ lines of $\mathscr{L}$ in a regulus.*

Proof We claim that a regulus R has exactly 2 rulings. If it had at least 3 rulings, then it contains 3 lines passing through every point of it, hence using the theory we developed in the section about flat points, we observe that each point of R is flat, so similar to the proof of lemma 3.14, we must have that a regulus is locally a plane, contradiction.

The previous lemma shows that if a regulus R contains more than 6 lines of $\mathscr{L}_p$, for some $p \in \mathscr{P}$, then all lines in one ruling of R must lie in $\mathscr{L}_p$. Hence there are at most 2 points $p_1, p_2 \in \mathscr{P}$ that contain more than 6 lines in R (as $\mathscr{L}_{p_1}$ and $\mathscr{L}_{p_2}$ disjoing by lemma 2.7. These two points contribute to at most $2N$ lines of $\mathscr{L}$ in R, while the other points contribute with at most $6N$. Therefore R contains at most $8N$ lines in $\mathscr{L}$.

So now we have showed $\mathscr{L}$ satisfies the hypothesis pf theorem 2.11. We are now ready to proceed with the proof. We will prove the following more general theorem:

Theorem 5.3 *Let $\mathscr{L}$ be a set of N^2 lines in $\mathbb{R}^3$ such that there are at most c_1N lines of $\mathscr{L}$ in any plane and at most c_2 lines of $\mathscr{L}$ in any regulus, for some constants c_1, c_2. Then there exist a constant $C(c_1, c_2)$ such that the number of points of intersections between any two lines in $\mathscr{L}$ is at most CN^3.*

We will show there exists a constant Q sufficiently large, depending only on c_1 and c_2, such that the number of points of intersections of lines in $\mathscr{L}$ is less than QN^3, for all N. We will give an explicit value for Q at the end of the proof.

We will proceed by induction on N. Clearly the statement if true for all $N \leq Q$, as there are at most N^4 distinct points of intersections between N^2 lines, and in this case $N^4 \leq QN^3$. For induction step, we suppose that for all $M < N$, if we have a collection of M^2 lines in $\mathbb{R}^3$ with at most c_1 of them in any plane and at most M^2 of them in any regulus, then there are at most QM^3 points of intersections. Suppose for contradiction $S = |I(\mathscr{L}, \mathscr{L})| \geq QN^3$, where $I(\mathscr{L}, \mathscr{L})$ denotes the set of intersections between lines in $\mathscr{L}$.

We would like to find a polynomial of bounded degree that contains most of the points of $I(\mathscr{L}, \mathscr{L})$. Note that if we were naive and using directly lemma 3.4, then we would obtain a polynomial of degree bounded by $6S^{1/3} \geq Q^{1/3}N$, which is too big. We would like to obtain a polynomial of degree $O(N)$. We will achieve this using a probabilistic method. First we will observe that it is enough to work with a subset of $\mathscr{L}$ consisting of lines with "many" intersection points.

Note that we have $\geq QN^3$ points of intersection between the N^2 lines of $\mathscr{L}$. Hence, the for a line in $\mathscr{L}$, the expected number number of distinct intersection points with other lines in $\mathscr{L}$ is $\gtrsim QN$. Denote by $\mathscr{L}'$ the subset of $\mathscr{L}$ consisting of lines which intersect other lines of $\mathscr{L}$ in at least $\frac{QN}{4}$ points. Hence

$$|I(\mathscr{L} \setminus \mathscr{L}', \mathscr{L})| \leq \frac{QN^3}{4} \ ,$$

so the lines in $\mathscr{L}'$ participate in at least $\frac{3QN^3}{4}$ points of intersection. Next denote $\mathscr{L}''$ the subset of $\mathscr{L}'$ consisting of lines which intersect other lines of $\mathscr{L}'$ in at least $\frac{QN}{8}$ points. Hence

$$|I(\mathscr{L}' \setminus \mathscr{L}'', \mathscr{L}')| \leq \frac{QN}{8}|\mathscr{L}'| \leq \frac{QN^3}{8} \ .$$

So

$$|I(\mathscr{L}'', \mathscr{L}'')| \geq |I(\mathscr{L}, \mathscr{L})| - |I(\mathscr{L} \setminus \mathscr{L}', \mathscr{L})| - |I(\mathscr{L}' \setminus \mathscr{L}'', \mathscr{L}')| \geq \frac{QN^3}{2} \ . \tag{14}$$

Thus we have obtained a subset $\mathscr{L}''$ of $\mathscr{L}$ such that each line in $\mathscr{L}''$ contains "many" intersection points with other lines in $\mathscr{L}''$ and $I(\mathscr{L}'', \mathscr{L}'')$ consists of most of the points in $I(\mathscr{L}, \mathscr{L})$. Say $|\mathscr{L}''| = \alpha N^2$. Now we are ready to apply the probabilistic argument. We need the following lemma.

Lemma 5.4 *Let $X \subset [n]$ a random subset such that each element of $[n]$ is included in X independently with probability p. Then we have*

1. $$\mathbb{P}(|X| \geq 2pn) \leq \exp\left(-\frac{pn}{4}\right)$$

2. $$\mathbb{P}\left(|X| \leq \frac{1}{2}pn\right) \leq \exp\left(-\frac{pn}{4}\right)$$

Proof Let

$$X_i = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{otherwise} \end{cases}$$

Then X_i are independent Bernoulli random variables with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1-p$. We can easily see that $|X| = X_1 + \dots X_n$. So we obtain

$$\mathbb{E}\left(e^{\alpha}|X|\right) = \mathbb{E}\left(\prod_{i=1}^{n} e^{\alpha X_i}\right) = \prod_{i=1}^{n} \mathbb{E}\left(e^{\alpha X_i}\right) = (pe^{\alpha} + 1 - p)^n \tag{15}$$

Also, Markov's inequality implies that if $\alpha > 0$, then

$$\mathbb{P}(|X| \geq 2pn) \leq \frac{\mathbb{E}\left(e^{\alpha}|X|\right)}{e^{2\alpha pn}}$$

Combining the previous 2 relations and letting $\alpha = 1$, we obtain

$$\mathbb{P}(|X| \geq 2pn)e^{2pn} \leq (pe + 1 - p)^n e^{-2pn}$$

Therefore, in order to prove the first relation, is is enough to show that, for all $0 \leq p \leq 1$,

$$\frac{pe + 1 - p}{e^{2p}} \leq \exp\left(-\frac{p}{4}\right) \ ,$$

which is easy to check.

Foe the second part, whenever $\alpha < 0$, we have that

$$\mathbb{P}(|X| \leq (1/2)pn)\ e^{2\alpha pn} \leq \mathbb{E}\left(e^{\alpha}|X|\right) \leq (pe^{\alpha} + 1 - p)^n$$

Take $\alpha = -\frac{1}{4}$, so it is enough to check that, for all $0 \leq p \leq 1$,

$$\frac{pe^{-1/4} + 1 - p}{e^{-p/2}} \leq \exp\left(-\frac{p}{4}\right) \ ,$$

which is easily verified. ∎

Let's return to our setting. We form a subset $\mathscr{L}_0$ of $\mathscr{L}''$ by including each line from $\mathscr{L}''$ independently with probability $\frac{16}{Q}$. Using the lemma above,

$$\mathbb{P}\left(|\mathscr{L}_0| \geq \frac{32\alpha N^2}{Q}\right) \leq \exp\left(-\frac{4\alpha N^2}{Q}\right) \leq \exp(-4) \leq 0.02$$

Here we used that $N \geq Q$ and that $|\mathscr{L}''| \geq N$ (since otherwise $|I(\mathscr{L}'', \mathscr{L}'')| \leq N^2$, contradiction with $|I(\mathscr{L}'', \mathscr{L}'')| \geq \frac{QN^3}{2}$). So we can assume $|\mathscr{L}_0| \leq \frac{32\alpha N^2}{Q}$.

Recal now that lemma 3.5 stated that for a set B of b lines in $\mathbb{R}^3$, we can find a non-zero polynomial of degree $\leq 4b^{1/2}$ which vanishes on all lines in B. Apply this lemma to our setting to obtain a non-zero polynomial P of degree $\leq \frac{24\sqrt{\alpha}N}{\sqrt{Q}}$ which vanishes on every line of $\mathscr{L}_0$.

Now fix a line l in $\mathscr{L}''$. It contains $\geq \frac{QN}{8}$ intersection points with with other lines $\mathscr{L}''$. Each of these points has a probability of at least $\frac{16}{Q}$ of lying in a line of $\mathscr{L}_0 \setminus \{l\}$, and the events are independent. We denote by T the number of distinct points of intersections of l with lines in $\mathscr{L}_0$. So by applying again lemma 5.4, we obtain that

$$\mathbb{P}(T \leq N) \leq \exp\left(-\frac{N}{2}\right) \ .$$

So the probability that there is at least one line in $\mathscr{L}''$ with less than N distinct points of intersections in $\mathscr{L}_0$ is at most $\exp(-N/2)\, N^2 \le 1/2$ as long as $N \ge 20$. So we find a choice of $\mathscr{L}_0$ such that $|\mathscr{L}_0| \le \frac{32\alpha N^2}{Q}$ and each line in $\mathscr{L}''$ has at least N intersections with lines in $\mathscr{L}_0$. As $N > \frac{24\sqrt{\alpha}N}{\sqrt{Q}}$, it means that the polynomial P vanishes on every line in $\mathscr{L}''$, so on at least $\frac{QN^3}{2}$ of the points of intersection in $\mathscr{L}$.

So we have found a polynomial of degree $O\left(\frac{N}{\sqrt{Q}}\right)$ which vanishes on at least $\frac{QN^3}{2}$ points of intersection. We would like to results about the geometry of ruled surfaces we have developed in the section 3.3 to obtain a contradiction. Factor $P = P_1 P_2$, where P_1 is the product of irreducible ruled factors of P and P_2 the product of of the other irreducible factors. Let d_1, d_2 be the degrees of P_1 and P_2 respectively. Denote by $\mathscr{L}_1 = Z(P_1) \cap \mathscr{L}''$ and $\mathscr{L}_2 = \mathscr{L}'' \setminus \mathscr{L}_1$.

Note that each for any $l \in \mathscr{L}_2$, $l \cap Z(P_1) \le d_1$, since $l \notin Z(P_1)$. So the number of intersections between lines of $\mathscr{L}_1$ and $\mathscr{L}_2$ is at most

$$|I(\mathscr{L}_1, \mathscr{L}_2)| \le d_1 |\mathscr{L}_2| \le \frac{24\sqrt{\alpha}N}{\sqrt{Q}} N^2 \le N^3 \ .$$

So thee must be at least $\frac{QN^3}{6}$ intersections either between the lines of $\mathscr{L}_1$ or between the lines of $\mathscr{L}_2$. We will treat each case separately.

First we'll show that lines in $\mathscr{L}_1$ cannot have many intersections. Similarly, we factor $P_1 = P_{11} P_{12}$, where P_{11} is the product of the irreducible factors which are not planes or reguli and P_{12} is the product of planes and reguli. Let $\mathscr{L}_{11} = \mathscr{L}_1 \cap Z(P_{11})$ and $\mathscr{L}_{12} = \mathscr{L}_1 \setminus \mathscr{L}_{11}$.

A line in $\mathscr{L}_{12}$ meets $Z(P_{11})$ in at most d_{11} points (where d_{11} is the degree of P_{11}), so similarly the number of intersections between between lines in $\mathscr{L}_{11}$ and $\mathscr{L}_{12}$ is at most N^3. Now, theorem 3.22 applied to P_{11} with $d = N$ implies that there are at most $4N^3$ intersection points between lines of $\mathscr{L}_{11}$. Lastly, suppose that P_{12} is the product of a_1 planes and a_2 reguli. We know there are at most $c_1 N$ lines in a plane, $c_2 N$ lines in a regulus. Also, it is easy to observe that a line has at most one intersection with a plane it is not contained in and at most 2 with a regulus it is not contained in. Hence, the number of intersections of lines in $\mathscr{L}_{12}$ is at most

$$|I(\mathscr{L}_{12}, \mathscr{L}_{12})| \le a_1 c_1^2 N^2 + a_2 c_2^2 N^2 + (a_1 + 2a_2) N^2 \le \frac{QN^3}{12} \ ,$$

for Q large enough (as the degree of P_{12} is $a_1 + 2a_2 \le N$).

So we must be in the second case. Recall that all lines of $\mathscr{L}_2$ belong to $Z(P_2)$, where P_2 is a polynomial of degree d_2 at most $\frac{24\sqrt{\alpha}N}{\sqrt{Q}}$. So using lemma 3.17, we obtain that $Z(P_2)$ must contain at most $11 d_2^2 \le \frac{10^4 \alpha N^2}{Q}$ lines. We would like to apply the induction hypothesis for the set of lines $\mathscr{L}_2$. Say $(M-1)^2 \le |\mathscr{L}_2| \le M^2$, where M is a positive integer. We can assume $|\mathscr{L}_2| = M^2$ (if not, just add some random lines to $\mathscr{L}_2$ until its size is a perfect square). So we can assume

$$M^2 = |\mathscr{L}_2| \le 2 \cdot 10^4 \cdot \frac{\alpha N^2}{Q} \ .$$

In order to apply the induction hypothesis, we would need to check that there are no more than $c_1 M$ lines in any plane $c_2 M$ lines in any regulus. If this would be true, then we would get that the number of intersections between in lines in $\mathscr{L}_2$ is at most

$$QM^3 \le 4 \cdot 10^6 \frac{N^3}{\sqrt{Q}}$$

which is clearly less than $\frac{QN^3}{6}$ for Q large enough.

If we are not lucky enough, then there are planes containing more than c_1M lines and reguli containing more than c_2M lines. If there is a plane containing more than c_1M lines of $\mathscr{L}_2$, we add them to a subset $\mathscr{L}_{21}$. Similar for a regulus containing more than c_2M lines of $\mathscr{L}_2$. So $\mathscr{L}_{21}$ consists of lines from at most $\frac{M}{c_1}$ planes and $\frac{M}{c_2}$ reguli. Similar as before, using that any plane must have of at most c_1N lines of $\mathscr{L}_{21}$, a regulus at most c_2N, and a line can have at most one intersection with a plane it is not contained in and at most with 2 a regulus, then the number of intersections of lines in $\mathscr{L}_{21}$ if at most

$$|I(\mathscr{L}_{21}, \mathscr{L}_{21})| \leq \frac{M}{c_1}(c_1N)^2 + \frac{M}{c_2}(c_2N)^2 + \left(\frac{M}{c_1} + 2\frac{M}{c_2}\right)|\mathscr{L}_{21}| \leq N^3 \ .$$

Denote by $\mathscr{L}_{22} = \mathscr{L}_2 \setminus \mathscr{L}_{21}$. Since no line in $\mathscr{L}_{22}$ belongs to a plane or regulus corresponding to $\mathscr{L}_{21}$, then the number of intersections between lines of $\mathscr{L}_{21}$ and $\mathscr{L}_{22}$ is at most

$$|I(\mathscr{L}_{21}, \mathscr{L}_{22})| \leq |\mathscr{L}_{22}|\left(\frac{M}{c_1} + 2\frac{M}{c_2}\right) \leq N^3 \ .$$

Now we are almost done. We know that there are at most c_1M lines of $\mathscr{L}_{22}$ in a plane and at most c_2M in a regulus. Just add some lines to $\mathscr{L}_{22}$ such that $|\mathscr{L}_{22}| = M^2$ and the number of lines in a plane or a regulus does not increase. So we can apply the induction hypothesis to obtain at most QM^3 intersections of lines in $\mathscr{L}_{22}$. This concludes that the second case must fail as well, so we have achieved a contradiction. So the proof is finally complete. ∎

Remark Note that we could take $Q = \max\{2 \cdot 10^4, 20c_1^2, 20c_2^2\}$ and all the estimates we used in the proof will hold.

6 Proof for $k \geq 3$

The goal of this section if to prove theorem 2.10. We will rely mainly on methods similar to those exposed in the section about the polynomial cell decomposition, but we will also use the geometry of flat and critical points. This suggest why it important that $k \geq 3$ separately, because a flat point of a polynomial P is defined to be a regular point of $Z(P)$ such that there are 3 distinct lines in $Z(P)$ intersecting at it. We will show that there for a given set $\mathscr{L}$ of lines, there exists a polynomial of relatively low degree that contains many of the lines in $\mathscr{L}$ and most of the points where at least k lines of $\mathscr{L}$ intersect and use the properties of this polynomial.

We will prove the following general result.

Theorem 6.1 *Let $\mathscr{L}$ be a finite set of L lines in $\mathbb{R}^3$ such that there are at most B lines in any plane. Let $\mathscr{P}$ be the set of points in $\mathbb{R}^3$ that belong to at least k lines in $\mathscr{L}$, for $k \geq 3$. Then*

$$|\mathscr{P}| = O(L^{3/2}k^{-2} + LBk^{-2} + Lk^{-1}) \ .$$

Remark Note that if we take $L = N^2$ and $B = N$, we easily obtain theorem 2.10.

6.1 Regularity adjustments

In this subsection we will show that we can impose some extra conditions and theorem 6.1 will still hold. We will first show that theorem 6.1 is equivalent to the following weaker form:

Theorem 6.2 *Let $\mathscr{L}$ be a finite set of L lines in $\mathbb{R}^3$ such that there are at most B lines in any plane. Let $\mathscr{P}$ be the set of points in $\mathbb{R}^3$ that intersect between k and $2k$ lines in $\mathscr{L}$, for $k \geq 3$. Then*

$$|\mathscr{P}| = O(L^{3/2}k^{-2} + LBk^{-2} + Lk^{-1}) \ .$$

Indeed, suppose that theorem 6.2 holds. Let $\mathscr{P}$ be the set of points that belong to at least k lines in $\mathscr{L}$. Denote by $k_j = 2^j k$ and $\mathscr{P}_j$ be the set of points that intersect between k_j and $2k_j$ lines in $\mathscr{L}$, for $j \geq 0$. Clearly,

$$\mathscr{P} \subset \bigcup_{j \geq 0} \mathscr{P}_j \ .$$

Also, since theorem 6.2 holds, there exists an universal constant C such that

$$|\mathscr{P}_j| \leq C(L^{3/2}k_j^{-2} + LBk_j^{-2} + Lk_j^{-1}) = C\left(\frac{1}{2^{2j}}L^{3/2}k^{-2} + \frac{1}{2^{2j}}LBk^{-2} + \frac{1}{2^j}Lk^{-1}\right) \ .$$

Hence

$$|\mathscr{P}_j| \leq \frac{1}{2^j}C(L^{3/2}k^{-2} + LBk^{-2} + Lk^{-1}) \ ,$$

so it follows that

$$|\mathscr{P}| \leq \sum_{j \geq 0} |\mathscr{P}_j| \leq \sum_{j \geq 0} \frac{1}{2^j}C(L^{3/2}k^{-2} + LBk^{-2} + Lk^{-1}) \leq 2C(L^{3/2}k^{-2} + LBk^{-2} + Lk^{-1}) \ .$$

Hence theorem 6.1 can easily be proved assuming theorem 6.2. From now on we work in the setting of theorem 6.2.

We would like to make an uniformity assumption about the lines. Note that there are $\sim |\mathscr{P}|k$ incidences between lines in $\mathscr{L}$ and points in $\mathscr{P}$. Therefore, the average points from $\mathscr{P}$ on a line from $\mathscr{L}$ is $\sim \frac{|\mathscr{P}|k}{L}$. We would like to know there are not too many lines in $\mathscr{L}$ which contain to few points of $\mathscr{P}$. Hence, we propose an even weaker form of 6.2:

Theorem 6.3 *Let $\mathscr{L}$ be a finite set of L lines in $\mathbb{R}^3$ such that there are at most B lines in any plane. Let $\mathscr{P}$ be the set of S points in $\mathbb{R}^3$ that intersect between k and $2k$ lines in $\mathscr{L}$, for some $k \geq 3$. Also, assume that there are at least $(1/8)L$ lines in $\mathscr{L}$ that contain $\geq (1/8)SkL^{-1}$ points of $\mathscr{P}$. Then*

$$S = O(L^{3/2}k^{-2} + LBk^{-2} + Lk^{-1}) \ .$$

We are now going to prove that theorems 6.3 and 6.2 are equivalent, which means we can assume the extra uniformity condition about the lines.

Lemma 6.4 *Theorem 6.3 implies theorem 6.2.*

Proof We are going to proceed by induction on the number of lines. Denote by $\mathscr{L}_1$ the subset of lines in $\mathscr{L}$ which contain at least $\frac{1}{8}\frac{Sk}{L}$ points of $\mathscr{P}$. If $|\mathscr{L}_1| \geq L/8$, by theorem 6.3, there exist a constant C such that

$$S \leq C(L^{3/2}k^{-2} + LBk^{-2} + Lk^{-1}) \ ,$$

which concludes the proof.

Hence suppose that $|\mathscr{L}_1| \leq L/8$. By assumption, we know that

$$Sk \leq |I(\mathscr{P}, \mathscr{L})| \leq 2Sk \ .$$

Also, it is easy to see that

$$|I(\mathscr{P}, \mathscr{L} \setminus \mathscr{L}_1)| \leq \frac{1}{8}\frac{Sk}{L}L = \frac{1}{8}Sk \ . \tag{16}$$

This means that lines in $\mathscr{L}_1$ contribute to most of the incidences. We would like to work with a subset of $\mathscr{P}$ such that each point belongs to "many" lines of $\mathscr{L}_1$. This suggests do define $\mathscr{P}_1 \subset \mathscr{P}$ the set of points with at least $(3/4)k$ incidences with lines of $\mathscr{L}_1$. So a point in $\mathscr{P} \setminus \mathscr{P}_1$ has at least $(1/4)k$ incidences with lines in $\mathscr{L} \setminus \mathscr{L}_1$ (as it belongs to at least k lines of $\mathscr{L}$). Then we observe the following inequalities:

$$\frac{1}{4}k|\mathscr{P} \setminus \mathscr{P}_1| \leq |I(\mathscr{P} \setminus \mathscr{P}_1, \mathscr{L} \setminus \mathscr{L}_1)| \leq \frac{1}{8}|\mathscr{P}|k \ ,$$

where for the last inequality we used (16). So we know

$$2|\mathscr{P} \setminus \mathscr{P}_1| \leq |\mathscr{P}| \ ,$$

hence it follows that

$$|\mathscr{P}_1| \geq \frac{1}{2}|\mathscr{P}|. \tag{17}$$

Now, a point in $\mathscr{P}_1$ belongs to between $(3/4)k$ and $2k$ points of $\mathscr{L}_1$. We would like to apply the induction hypothesis for $\mathscr{L}_1$, but we encounter a problem as the range of number of intersections is larger than we considered before. This can be easily fixed by splitting $\mathscr{P}_1$ into two parts. Let $\mathscr{P}_{11} \subset \mathscr{P}_1$ consist of the points which have between $(3/4)k$ and k incidences with lines in $\mathscr{L}_1$ and $\mathscr{P}_{12} \subset \mathscr{P}$ the points which have between k and $2k$ incidences with lines in $\mathscr{L}_1$. Clearly, $\mathscr{P}_1 = \mathscr{P}_{11} \cup \mathscr{P}_{12}$. Denote $\mathscr{P}_2$ the larger of the sets $\mathscr{P}_{11}$ and $\mathscr{P}_{12}$. Using (17), we note that

$$|\mathscr{P}_2| \geq (1/4)|\mathscr{P}| \ . \tag{18}$$

- If $\mathscr{P}_2 = \mathscr{P}_{11}$, define $k_2 = \lceil (3/4)k \rceil$. Then

$$k_2 \geq \lceil (3/4)k \rceil \geq \lceil (3/4) \cdot 3 \rceil \geq 3$$

- If $\mathscr{P}_2 = \mathscr{P}_{12}$, define $k_2 = k$.

So now each point in $\mathscr{P}_2$ has between k_2 and $2k_2$ incidences with lines in $\mathscr{L}_1$. By induction hypothesis, it follows that

$$|\mathscr{P}_2| \leq C(|\mathscr{L}_1|^{3/2}k_2^{-2} + |\mathscr{L}_1|Bk_2^{-2} + |\mathscr{L}_1|k_2^{-1}) \tag{19}$$

But now we use that $|\mathscr{P}_2| \geq (1/4)|\mathscr{P}| = (1/4)S$, $|\mathscr{L}_1| \leq L/8$ and $k_2 \geq (3/4)k$ to obtain

$$\begin{aligned} S \leq 4|\mathscr{P}_2| &\leq 4C\left(\left(\frac{L}{8}\right)^{3/2}\left(\frac{3}{4}k\right)^{-2} + \left(\frac{L}{8}\right)B\left(\frac{3}{4}k\right)^{-2} + \left(\frac{L}{8}\right)\left(\frac{3}{4}k\right)^{-1}\right) \\ &\leq \left(4 \cdot \frac{1}{8}\left(\frac{4}{3}\right)^2\right)C(L^{3/2}k^{-2} + LBk^{-2} + Lk^{-1}) \\ &\leq C(L^{3/2}k^{-2} + LBk^{-2} + Lk^{-1}) \end{aligned}$$

Hence, the proof is finished. ∎

Remark It is interesting to note that we obtain the same constant in theorem 6.2 as in the weaker theorem 6.3.

6.2 Rest of the proof

The goal of this chapter is to prove 6.3. Say $|\mathscr{P}| = S$. We want to show that there exists a universal constant C such that $S \leq C(L^{3/2}k^{-2} + LBk^{-2} + Lk^{-1})$. We will show that if

$$S \geq Q(L^{3/2}k^{-2} + Lk^{-1}) \tag{20}$$

for a constant Q large enough, then we will find a plane containg CSk^2L^{-1} lines, for some fixed constant C. This would imply that $S \leq (1/C) \cdot BLk^{-2}$, which implies 6.3. We will give an explicit value of Q and C for which the argument works.

We begin by showing that if (20) holds, then most of the points in $\mathscr{P}$ belong to the zero set of a polynomial of "low" degree. The polynomial cell decomposition methods will be essential for this.

Lemma 6.5 *Let $\mathscr{L}$ be a finite set of L lines in $\mathbb{R}^3$ and $\mathscr{P}$ be the set of S points that belong to at least k lines in $\mathscr{L}$. Also, assume that $S \geq 2^{31}(L^{3/2}k^{-2} + Lk^{-1})$. Then there exists a non-zero polynomial P of degree at most $2^{66}L^2S^{-1}k^{-3}$ such that $Z(P)$ contains at least $(1-2^{-10})S$ points of $\mathscr{P}$.*

Proof Let d be a parameter we will choose later. We apply the cell decomposition theorem 4.6 for d^3, so we find a non-zero polynomial P of degree at most $30d$ such that $\mathbb{R}^3 \setminus Z(P)$ can be partitioned into at most $2d^3$ cells, each cell containing at most S/d^3 points of $\mathscr{P}$. Denote the cells Ω_i, for $1 \leq i \leq m$, where $m \leq 2d^3$.

Suppose for contradiction $Z(P)$ contains less than $(1-2^{-10})S$ points of $\mathscr{P}$. This means there are at least $2^{-10}S$ points of $\mathscr{P}$ in the union of all cells.

Call the cells that contain at least $2^{-12}Sd^{-3}$ points of $\mathscr{P}$ full cells. Let $J \subset [m]$ be the set of indices of the full cells, i.e.

$$|\Omega_i \cap \mathscr{P}| \geq 2^{-12}Sd^{-3} \iff i \in J\ .$$

Then we have

$$2^{-10}S \leq \sum_{i=1}^{m} |\Omega_i \cap \mathscr{P}| \leq \sum_{i \in J} \frac{S}{d^3} + \sum_{i \notin J} \frac{1}{2^{12}}\frac{S}{d^3} \leq |J|Sd^{-3} + 2d^3 \cdot 2^{-12}Sd^{-3}\ .$$

It follows that $|J| \geq 2^{-11}d^3$, so there are at least $2^{-11}d^3$ full cells.

Denote $\mathscr{L}_i \subset \mathscr{L}$ be the subset of lines of $\mathscr{L}$ that intersect Ω_i. Let

$$L_0 = \min\{|\mathscr{L}_i| : i \in J\}$$

which is the minimum number of lines intersecting any of the full cells. We want to find an upper bound for L_0. In order to do this, look at the pairs

$$\{(l, \Omega_i) : l \in \mathscr{L}_i, i \in J\}$$

We note that any line intersects at most $d+1$ cells, because every time a line moves from one cell to another, it must intersect $Z(P)$. As we've seen earlier, if a line has more than d intersections with $Z(P)$, then it is included in $Z(P)$, so has zero intersections with any cell. Hence we obtain the following inequalities

$$L_0 \cdot 2^{-11}d^3 \leq L_0|J| \leq |\{(l, \Omega_i) : l \in \mathscr{L}_i, i \in J\}| \leq L(d+1) \leq 2Ld$$

so we obtain

$$L_0 \leq 2^{12}Ld^{-2} \ .$$

Next we apply Szémeredi-Trotter theorem (theorem 4.9) for the set of lines $\mathscr{L}_i$, where Ω_i is the full cell with the fewest lines. By assumption, each point in $\mathscr{P} \cap \Omega_i$ belongs to at least k lines in $\mathscr{L}_i$. So we obtain

$$2^{-12}Sd^{-3} \leq 2^{24}\left(\frac{L_0^2}{k^3} + \frac{L_0}{k}\right) \leq 2^{48}(L^2d^{-4}k^{-3} + Ld^{-2}k^{-1})$$

Hence

$$S \leq 2^{60}(L^2d^{-1}k^{-3} + Ldk^{-1}) \tag{21}$$

In order to get rid of the first term in the bracket, this suggests taking

$$d = 2^{61}L^2S^{-1}k^{-3} \ . \tag{22}$$

We first need to check that $d \geq 1$. Indeed, we know from Szémeredi - Trotter that

$$S \leq 2^{24}(L^2k^{-3} + Lk^{-1}) \ .$$

But, by assumption $S \geq 2^{31}(L^{3/2}k^{-2} + Lk^{-1}) \geq 2^{25}Lk^{-1}$, so $2^{24}Lk^{-1} \leq S/2$, so we must have $S \leq 2^{25}L^2k^{-3}$. Therefore,

$$1 \leq 2^{25}L^2S^{-1}k^{-3} \leq 2^{61}L^2S^{-1}k^{-3} = d \ .$$

Now, putting together (21) and (22), we obtain

$$S/2 \leq 2^{60}L^3S^{-1}k^{-4}$$

which implies that

$$S < 2^{31}L^{3/2}k^{-2} \ ,$$

contradiction with the assumption. ∎

So far we've shown that if $S \geq Q(L^{3/2}k^{-2} + Lk^{-1})$, for a constant Q large enough, then most of the points in $\mathscr{P}$ will belong to the zero set of a polynomial of degree $O(L^2S^{-1}k^{-3})$. Denote by $\mathscr{P}_1 = \mathscr{P} \cap Z(P)$ and by $\mathscr{L}_1$ the set of lines in $\mathscr{L}$ that are contained in $Z(P)$. We would like to show that many of the lines in $\mathscr{L}$ belong to $\mathscr{L}_1$. First, we will check that the degree d of P is small enough compared to SkL^{-1} (which is the approximately the average number of points in $\mathscr{P}$ on a line in $\mathscr{L}$), if we assume Q in (20) is large enough.

Lemma 6.6 $d < 2^{-12}SkL^{-1}$.

Proof Indeed, using the estimates in the previous lemma, it is enough to verify that

$$2^{66}L^2S^{-1}k^{-3} \leq 2^{-12}SkL^{-1} \iff S^2 \geq 2^{78}L^3k^{-4} \iff S \geq 2^{39}L^{3/2}k^{-2}$$

which is true by (20) if Q large enough.

Now we are ready to show that an important ratio of the lines in $\mathscr{L}$ lie in $Z(P)$. The uniformity assumptions will turn out to be essential in order to achieve this.

Lemma 6.7 $|\mathscr{L}_1| \geq (1/16)L$.

Proof By assumption, there are at least $(1/8)L$ lines in $\mathscr{L}$ such that each contain at least $(1/8)SkL^{-1}$ points of $\mathscr{P}$. Call the set of such lines $\mathscr{L}_0$. We would like to show that most of these lines belong to $\mathscr{L}_1$.

We note that a line in $\mathscr{L}_0 \setminus \mathscr{L}_1$ contains at least $(1/8)SkL^{-1}$ points of $\mathscr{P}$, but has less than $2^{-12}SkL^{-1}$ with points of $\mathscr{P}_1$ (a line which does not belong to $Z(P)$ has at most d intersections with $Z(P)$). So it must have at least $(1/16)SkL^{-1}$ points of $\mathscr{P} \setminus \mathscr{P}_1$. Hence

$$(1/16)SkL^{-1}|\mathscr{L}_0 \setminus \mathscr{L}_1| \leq |I(\mathscr{P} \setminus \mathscr{P}_1, \mathscr{L}_0 \setminus \mathscr{L}_1)| \leq 2k|\mathscr{P} \setminus \mathscr{P}_1| \leq 2k \cdot 2^{-10}S \ .$$

For the middle inequality we used the assumption that each point in $\mathscr{P}$ belongs to at most $2k$ lines of $\mathscr{L}$.

Therefore we have that

$$|\mathscr{L}_0 \setminus \mathscr{L}_1| \leq (1/16)L \ .$$

This together with out assumption that $|\mathscr{L}_0| \geq (1/8)L$ imply that $|\mathscr{L}_0 \cap \mathscr{L}_1| \geq (1/16)L$. ∎

Let's recapitulate what we have achieved so far: we have constructed a polynomial of degree $O(L^2S^{-1}k^{-3})$ that contains at least $(1 - 2^{-10})S$ points of $\mathscr{P}$ and at least $(1/16)L$ lines in $\mathscr{L}$. We want to use the theory of critical and flat points. Recall that a point in $Z(P)$ is flat if it is not critical and belongs to at least 3 different lines in $Z(P)$. Therefore it is natural to define the subset $\mathscr{P}_2 \subset \mathscr{P}_1$ of points that belong to at least 3 different lines in $\mathscr{L}_1$. Naturally, as before, we would like to show that most points in $\mathscr{P}$ belong to $\mathscr{P}_2$.

Lemma 6.8 $|\mathscr{P} \setminus \mathscr{P}_2| \leq 2^{-9}S$.

Proof We note that a point in $\mathscr{P}_1 \setminus \mathscr{P}_2$ lies in at least k lines of $\mathscr{L}$, but at most 2 lines of $\mathscr{L}_1$, so it belongs to at least $k-2$ lines in $\mathscr{L} \setminus \mathscr{L}_1$. Hence the following inequality holds:

$$(k-2)|\mathscr{P}_1 \setminus \mathscr{P}_2| \leq |I(\mathscr{P}_1 \setminus \mathscr{P}_2, \mathscr{L} \setminus \mathscr{L}_1)| \leq (2^{-12}SkL^{-1}) \cdot L$$

For the second inequality, we used that a line in $\mathscr{L} \setminus \mathscr{L}_1$ intersects $Z(P)$ at most d times, and that $d \leq 2^{-12}SkL^{-12}$, as seen in lemma 6.6.

Hence

$$|\mathscr{P}_1 \setminus \mathscr{P}_2| \leq 2^{-12}\frac{k}{k-2}S \leq 3 \cdot 2^{-12}S \leq 2^{-10}S$$

We used, of course, that $k \geq 3$. Recall from lemma 6.5 that $|\mathscr{P} \setminus \mathscr{P}_1| \leq 2^{-10}S$ and that $\mathscr{P}_2 \subset \mathscr{P}_1 \subset \mathscr{P}$, so we have

$$|\mathscr{P} \setminus \mathscr{P}_2| = |\mathscr{P} \setminus \mathscr{P}_1| + |\mathscr{P}_1 \setminus \mathscr{P}_2| \leq 2^{-9}S \ . \quad ∎$$

We would like to show that there are many lines in $\mathscr{L}$ that contain "many" points of $\mathscr{P}_2$. Define $\mathscr{L}_2 \subset \mathscr{L}$ be a subset of lines of $\mathscr{L}$ such that each line contains at least $(1/16)SkL^{-1}$ points of $\mathscr{P}_2$.

Lemma 6.9 $|\mathscr{L}_2| \geq L/16$.

Proof Similar as in the proof of lemma 6.7, a line in $\mathscr{L}_0 \setminus \mathscr{L}_2$ contains at least $(1/8)SkL^{-1}$ points of $\mathscr{P}$, but at most $2^{-12}SkL^{-1}$ points of $\mathscr{P}_2$. Hence

$$(1/16)SkL^{-1}|\mathscr{L}_0 \setminus \mathscr{L}_2| \leq |I(\mathscr{P} \setminus \mathscr{P}_2, \mathscr{L} \setminus \mathscr{L}_2)| \leq 2k|\mathscr{P} \setminus \mathscr{P}_2| \leq (2k) \cdot 2^{-9}S$$

This implies that

$$|\mathscr{L}_0 \setminus \mathscr{L}_2| \leq (1/16)L.$$

Also, from assumption, we know that $|\mathscr{L}_0| \geq L/8$. Combining the last two relations, we obtain the desired result. ∎

So far we have shown there exists there exists a polynomial of degree $O(L^2S^{-1}k^{-3})$ such that the subset $\mathscr{P}_2 \subset \mathscr{P}$ of points of $\mathscr{P}$ belonging to at least 3 lines in $Z(P) \cap \mathscr{L}$ is large enough, in the sense that $|\mathscr{P} \setminus \mathscr{P}_2| \leq 2^{-9}|\mathscr{P}|$. Also we have shown that there are many lines in $\mathscr{L}$ containing at least $(1/16)SkL^{-1}$ points of $\mathscr{P}_2$. We are now ready to apply the results from section 3.2 about critical and flat points. In order to do this, we will assume that P is square-free. Note that by removing the repeating irreducible factors of P, then $Z(P)$ remains invariant and the degree of P reduces. So we can make this assumption without losing our generality.

Lemma 6.10 *Every line in $\mathscr{L}_2$ is either critical or flat.*

Proof By definition, every point in $\mathscr{P}_2$ is either critical or flat. Also, each line in $\mathscr{L}_2$ contains at least $(1/16)SkL^{-1}$ points of $\mathscr{P}_2$. So each line in $\mathscr{L}_2$ contains either $(1/32)SkL^{-1}$ critical points or $(1/32)SkL^{-1}$ flat points. Recall lemmas 3.7 and 3.13, which stated that if a line contains more than d critical points or more than $3d-4$ flat points, then the line is critical or flat respectively. Note that for the lemma 3.13 to hold, we need to have all lines in $\mathscr{L}_2$ in general position. We can achieve this easily by a change of coordinates (a ratation of the coordinate frame will do). Combining this with that fact that $d < 2^{-12}SkL^{-1}$ gives the required result. ∎

We note that lemma 3.9 implies there are at most d^2 critical lines. We would like to show that d^2 is very small compared to L, and this would imply that most lines of $\mathscr{L}_2$.

Lemma 6.11 $d \leq (1/16)L^{1/2}$.

Proof We know by assumption (20) that $S \geq QL^{3/2}k^{-2}$, so $1 \leq Q^{-1}SL^{-3/2}k^2$. Using our estimates from 6.5, we obtain

$$d \leq 2^{66}L^2S^{-1}k^{-3} \leq (2^{66}L^2S^{-1}k^{-3})(Q^{-1}SL^{-3/2}k^2) \leq 2^{66}Q^{-1}L^{1/2}k^{-1} \leq (1/16)L^{1/2}$$

if we assume Q is large enough. ∎

This means that there are at most $d^2 \leq (1/256)L$ critical lines. Since $\mathscr{L}_2 \geq (1/16)L$ and each line in $\mathscr{L}_2$ is either critical or flat, it follows that there are at least $(1/32)L$ flat lines in $\mathscr{L}_2$. Recall that lemma 3.15 says that a square-free polynomial P' with no factors of degree 1 has at most $3d'^2$ flat lines, where $d' = \deg(P')$. This suggests to factor our polynomial $P = P_1P_2$, where P_1 is the product of linear factors of P, and P_2 the product of irreducible factors of P of degree at least 2.

Note that since $\nabla P = \nabla P_1 P_2 + P_1 \nabla P_2$, it follows that P_1 and P_2 don't vanish simultaneously on a flat line l of P (since otherwise ∇P vanish on l, hence l is critical). So an argument from the proof of lemma 3.15 shows that if l is a flat line for P, then it must be a flat line for one of P_1 or P_2. Indeed, say $a \in l$ is a flat point for P such that $P_1(a) = 0$ and $P_2(a) \neq 0$. Then a belongs to 3 lines in $Z(P)$, so these lines must belong to $Z(P_1)$ (because locally around a, we have $P_2 \neq 0$), hence a is a flat point for P_1.

Now, if we apply lemma 3.15 to the polynomial P_2, we obtain that P_2 has at most $3d^2 \leq (3/256)L$ flat lines. This means that $\mathscr{L}_2$ contains at least $(1/64)L$ flat lines of P_1. But $Z(P_1)$ is a union of $O(L^2S^{-1}k^{-3})$ planes. So we obtain that many of the lines in $\mathscr{L}$ belong to a union of planes. So we can obtain a bound for the number of lines in a plane, we're almost finished.

We know $d \leq 2^{66}L^2S^{-1}k^{-3}$ and there are at least $(1/64)L$ lines belonging to at most d planes. So there must be a plane containing at least $2^{-72}Sk^3L^{-1}$ lines, so

$$B \geq 2^{-72}Sk^3L^{-1} \ .$$

This implies that

$$S \leq 2^{72}BLk^{-3} \ .$$

Recall what we started with: we assumed that $S \geq Q(L^{3/2}k^{-2} + Lk^{-1})$, for a constant large enough, and we deduced that $S \leq c_1BLk^{-3}$, for some constant $c_1 \leq 2^{72}$. Hence

$$S \leq \max\{c_1, Q\}(L^{3/2}k^{-2} + BLk^{-3} + Lk^{-1}) \ .$$

Note that we can take $Q = 2^{40}$ and all the proofs will work.

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