# Distribution of Modular Symbols

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- $\Gamma_0(N)$ acts on $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \infty$. We denote by $X_0(N)$ the quotient surface $\Gamma_0(N) \backslash \mathbb{H}^*$.

# Modular forms

## Definition

A *cusp form* of weight $k$ (and level $N$) is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

- For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have that
$$f(\gamma z) = (cz + d)^k f(z) \ ;$$
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We note that if $f \in S_2(\Gamma_0(N))$, then $f(z)dz$ is a $\Gamma_0(N)$-invariant cuspidal 1-form.

# Modular symbols

The cusps for $\Gamma_0(N)$ are parametrized by $\mathbb{Q}$, so for $r \in \mathbb{Q}$ and $f \in S_2(\Gamma_0(N))$, we define the modular symbol

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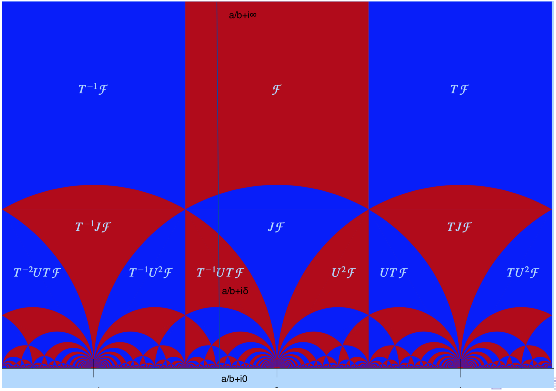

# Central values of $L$-functions

Let $f \in S_k(\Gamma)$ be a cusp form of weight $k$ with Fourier expansion at $\infty$

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$$L(f \otimes e(r), s) := \sum_{n \geqslant 1} \frac{a_f(n) e(nr)}{n^s}, \quad \text{for } \operatorname{Re}(s) > 1 \ .$$

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- Central value: $\langle r \rangle = L(f \otimes e(r), 1/2)$

# The Birch–Stevens formula

Let $E/\mathbb{Q}$ be an elliptic curve with associated weight 2 holomorphic cusp form $f(z)$. Let $\chi$ be a primitive character mod $c$. Then

$$\tau(\chi)L(E,\bar{\chi},1) = \sum_{a\in(\mathbb{C}/c\mathbb{Z})^\times} \bar{\chi}(a)\left\langle \frac{a}{c} \right\rangle$$

# Homology

- Let $A, B \in \mathbb{H}^*$ which are $\Gamma$-equivalent, i.e. $\exists \gamma \in \Gamma$ such that $B = \gamma(A)$. Any smooth path from $A$ to $B$ determines a unique homology class in $H_1(X_0(N), \mathbb{Z})$. We denote this homology class by the 'modular symbol' $\{A, B\}$.

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- We can extend the definition of $\{A, B\}$ to points $A, B \in \mathbb{H}^*$ not necesarilly $\Gamma$-equivalent by identifying $\{A, B\} \in H_1(X_0(N), \mathbb{C})$ with the functional $f \mapsto \int_A^B f(z)dz$.

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Integration defines a pairing

$$\langle , \rangle : S_2(\Gamma_0(N)) \times H_1(X_0(N), \mathbb{Z}) \to \mathbb{C}$$

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**Theorem**

*We have an induced perfect pairing*

$$\langle , \rangle : S_2(\Gamma_0(N)) \times H_1(X_0(N), \mathbb{R}) \to \mathbb{C}$$

# Mazur-Rubin-Stein conjecture

For $f \in SL_2(\Gamma_0(N))$, we have that $f(z+1) = f(z)$. From now on we work with the real-valued modular symbol

$$\langle r \rangle = \int_{i\infty}^{r} \mathrm{Re}(f(z)dz).$$

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## Conjecture (Mazur-Rubin-Stein)

*Fix* $x \in [0,1]$ *and let*

$$G_c(x) = \frac{1}{c} \sum_{0 \leqslant a/c \leqslant x} \left\langle \frac{a}{c} \right\rangle$$

*Then*

$$\lim_{c \to \infty} G_c(x) = \sum_{n=1}^{\infty} \frac{\mathrm{Re}(a(n)(e(nx)-1))}{n^2} = g(x)$$

# Numerics

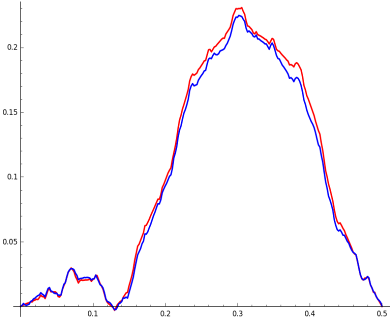


Figure: Plot of $G_c(x)$, $c = 1009$, for $E = 11a$

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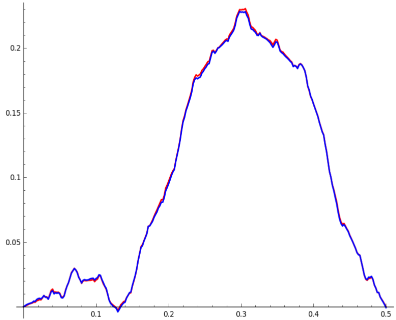


Figure: Plot of $G_c(x)$, $c = 10007$, for $E = 11a$

# Recent progress

**Theorem (Petridis–Risager, 2017)**

*For $\Gamma_0(q)$, $q$ squarefree. For all $x \in [0,1]$:*

$$\frac{1}{M}\sum_{c \leqslant M} G_c(x) \to g(x)$$

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**Theorem (Diamantis–Hoffstein–Kiral–Lee, 2018)**

*For all $q$,*

$$G_c(x) = g(x) + O(c^{-1/4} q^{1/4} (cq)^{\epsilon})$$

# Expectation and Variance

$$E(f,c) := \frac{1}{\phi(c)} \sum_{\substack{a \bmod c \\ (a,c)=1}} \langle a/c \rangle \quad Var(f,c) := \frac{1}{\phi(c)} \sum_{\substack{a \bmod c \\ (a,c)=1}} \left(\langle a/c \rangle - E(f,c)\right)^2$$

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## Conjecture (Mazur-Rubin)

*There exists a constant $C_f$ and constants $D_{f,d}$ for each divisor $d$ of $q$ such that*

$$\lim_{\substack{c \to \infty \\ (c,q)=d}} (Var(f,c) - C_f \log c) = D_{f,d}$$

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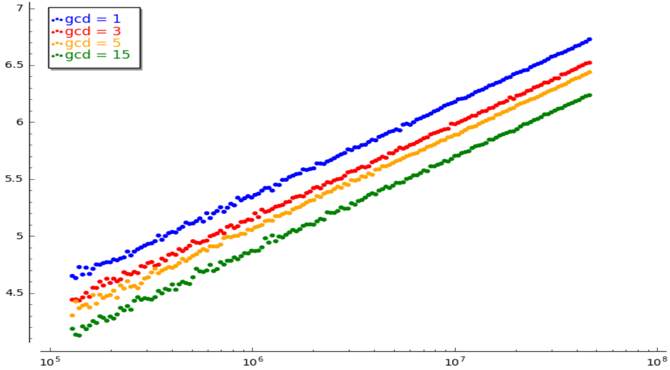
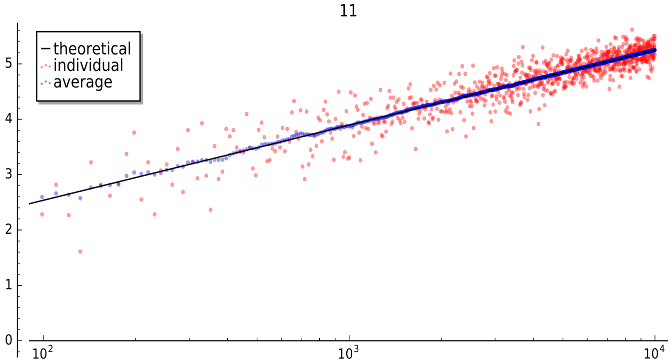
## Theorem (Petridis–Risager, 2017)

$$\frac{1}{\sum_{\substack{c \leqslant M \\ (c,q)=d}} \phi(c)} \sum_{\substack{c \leqslant M \\ (c,q)=d}} \phi(c)(Var(f,c) - C_f \log c) \to D_{f,d}, \quad M \to \infty$$

# Plot of $\mathrm{Var}(E, c)$ for $E = 15A1$, $\gcd(c, 15) = d$



# Plot of $\mathrm{Var}(E, c)$ for $E = 11a1$ and $\gcd(c, 11) = 11$



# Normal distribution for modular symbols

**Theorem (Petridis-Risager, 2017, after a conjecture of Mazur–Rubin)**

*$I \subseteq \mathbb{R}/\mathbb{Z}$ interval with $\lambda(I) > 0$.*
*For $d|q$ set*

$$Q_d = \{a/c \in \mathbb{Q}, (a,c) = 1, (c,q) = d\}.$$

*Then the values of*

$$\begin{array}{ccc} Q_d \cap I & \to & \mathbb{R} \\ \dfrac{a}{c} & \mapsto & \dfrac{\langle a/c \rangle}{(C_f \log c)^{1/2}}, \end{array}$$

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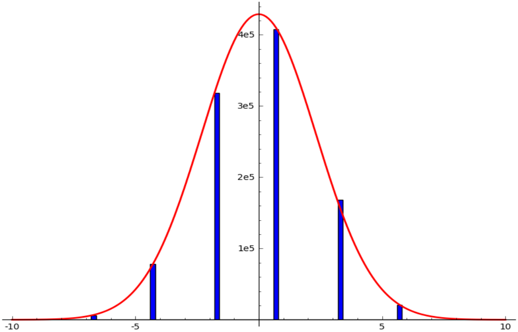
*have limit the standard normal distribution, i.e. as $X \to \infty$*

$$\frac{\left|\left\{\frac{a}{c} \in I, 0 < c < X, \frac{\langle a/c \rangle}{\sqrt{C_f \log c}} \in [A,B]\right\}\right|}{\left|\left\{\frac{a}{c} \in I, 0 < c < X\right\}\right|} \to \frac{1}{\sqrt{2\pi}} \int_A^B e^{-x^2/2} dx$$

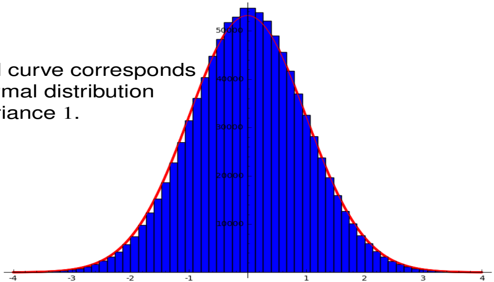
# Histogram of normalized modular symbols

Histogram of $\{[a/m]_E^+ : E = 11A1, m = 1,000,003, a \in (\mathbb{Z}/m\mathbb{Z})^\times\}$



## $E = 11a1$

The red curve corresponds to a normal distribution with variance $1$.



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$$E_{\mathfrak{a}}(z, s, \epsilon) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\chi_\epsilon(\gamma)} \, \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s, \quad \mathrm{Re}(s) > 1$$

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# Selberg's theory of twisted Eisenstein series

- Fourier expansion

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \epsilon) = \delta_{\mathfrak{a}\mathfrak{b}} y^{s} + \phi_{\mathfrak{a}\mathfrak{b}}(s, \epsilon) y^{1-s} + \sum_{n} \phi_{\mathfrak{a}\mathfrak{b}}(s, n, \epsilon) \sqrt{y} K_{it}(2\pi |n| y) e(nx)$$

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- Let $\sigma$ be such a pole, then denote $u_{\mathfrak{a},\sigma}(z, \epsilon) = \mathrm{Res}_{s=\sigma} E_{\mathfrak{a}}(z, s, \epsilon)$
- $u_{\mathfrak{a},\sigma} \in L^2(\Gamma \backslash \mathbb{H}, \chi_\epsilon)$ and it satisfies $(\Delta + \sigma(2 - \sigma)) u_{\mathfrak{a},\sigma}(\cdot, \epsilon) = 0$.

# Spectral Theory of $L^2(\Gamma\backslash\mathbb{H}, \chi_\epsilon)$

- Denote by $L^2(\Gamma\backslash\mathbb{H}, \chi_\epsilon)$ the space of square integrable functions on $\Gamma\backslash\mathbb{H}$ with respect to the hyperbolic metric, satisfying $f(\gamma z) = \chi_\epsilon(\gamma) f(z)$.

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**Conjecture (Selberg's eigenvalue conjecture)**

$\lambda_1(0) \geqslant 1/4.$

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**Theorem (Berry–Essen)**

*If $\mathcal{Y}$ is a real valued random variable and $T > 0$, then*

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- Large $t$: Both $e^{-t^2/2}$ and $\mathbb{E}(\exp(it\mathcal{Y}))$ are 'very small'

# Modular symbols on $\mathbb{H}^3$

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| $\langle r \rangle = \int_{i\infty}^{r} \mathrm{Re}(f(z)dz)$, $r \in \mathbb{Q}$ | $\langle r \rangle = \int_{j\infty}^{r} \mathrm{Re}(F.\beta)$, $r \in K$ |

# Distribution of modular symbols in $\mathbb{H}^3$

**Theorem (C., 2019)**

*Let $K$ be a quadratic imaginary field of class number one and $\mathfrak{n} \lhd \mathcal{O}_K$ a square-free ideal with generator $\langle n \rangle = \mathfrak{n}$. For $\mathfrak{b} | \mathfrak{n}$, set*

$$Q_{\mathfrak{b}}(X) = \{a/c \mid a \in (\mathcal{O}_K/\langle c \rangle)^\times, \langle c, n \rangle = \mathfrak{b}, 0 < |c| < X\}.$$

*Let $F \in S_2(\Gamma_0(\mathfrak{n}))$. Then the data*

$$K \cap Q_{\mathfrak{b}}(X) \to \mathbb{R} \quad \frac{a}{c} \mapsto \frac{\langle a/c \rangle}{\sqrt{C_F \log X}}$$

*has asymptotically a standard normal distribution.*

# Distribution of modular symbols in $\mathbb{H}^3$

## Theorem (C., 2019)

*Let $K$ be a quadratic imaginary field of class number one and $\mathfrak{n} \lhd \mathcal{O}_K$ a square-free ideal with generator $\langle n \rangle = \mathfrak{n}$. For $\mathfrak{b}|\mathfrak{n}$, set*

$$Q_{\mathfrak{b}}(X) = \{a/c \mid a \in (\mathcal{O}_K/\langle c \rangle)^{\times}, \langle c, n \rangle = \mathfrak{b}, 0 < |c| < X\}.$$

*Let $F \in S_2(\Gamma_0(\mathfrak{n}))$. Then the data*

$$K \cap Q_{\mathfrak{b}}(X) \to \mathbb{R} \quad \frac{a}{c} \mapsto \frac{\langle a/c \rangle}{\sqrt{C_F \log X}}$$

*has asymptotically a standard normal distribution.*
*Also, there exists a constant $D_{F,\mathfrak{b}}$ such that*

$$\frac{1}{Q_{\mathfrak{b}}(X)} \sum_{\substack{|c| \leqslant X \\ \langle c,n \rangle = \mathfrak{b}}} |(\mathcal{O}_K/\langle c \rangle)^{\times}|(Var(F,c) - C_F \log c) \to D_{F,d}$$